

Lindeberg's central limit theorem for martingale like sequences under nonlinear expectations*

LI-XIN ZHANG¹

Department of Mathematics, Zhejiang University, Hangzhou 310027

(Email:stazlx@zju.edu.cn)

Abstract

General central limit theorems and functional central limit theorems are obtained for martingale like random variables under the sublinear expectation. As applications, the Lindeberg central limit theorem and functional central limit theorem are obtained for independent but not necessarily identically distributed random variables, and a new proof of the Lévy characterization of a G-Brownian motion without using stochastic calculus is given. For proving the results, we have also established Rosenthal's inequality and the exceptional inequality for the martingale like random variables.

Keywords: sub-linear expectation; capacity; central limit theorem; functional central limit theorem; martingale difference.

AMS 2010 subject classifications: 60F15; 60F05

1 Introduction and notations.

Non-additive probabilities and non-additive expectations are useful tools for studying uncertainties in statistics, measures of risk, superhedging in finance and non-linear stochastic calculus, cf. Denis and Martini (2006), Gilboa (1987), Marinacci (1999), Peng (1997, 1999, 2006, 2007b, 2008a) etc. Peng (2006) introduced the notion of sublinear expectation. Under the sublinear expectations, Peng (2006, 2007a, 2007b, 2008a, 2008b, 2009) gave the notions of G-normal distributions, G-Brownian motions, G-martingales, independence of random variables, identical distribution of random variables and so on, and developed the weak law of large numbers and central limit theorem for independent and identically distributed (i.i.d.) random variables. Furthermore, Peng established the stochastic calculus with respect to the G-Brownian motion. As a result, Peng's framework of nonlinear expectation gives a generalization of Kolmogorov's probability theory. Recently, Bayraktar and Munk

¹Research supported by Grants from the National Natural Science Foundation of China (No. 11225104).

(2016) proved an α -stable central limit theorem for independent and identically distributed random variables.

This paper considers the general central limit theorem for random variables which are not necessarily i.i.d. We establish a central limit theorem and a functional central limit theorem for a kind of martingale-difference like random variables under the conditional Lindeberg condition. As applications, we establish the central limit theorem and functional central limit theorem for independent but not necessary identically distributed under the popular Lindeberg's condition. The tool for proving the central limit theorem is a promotion of Peng (2008b)'s and gives also a new normal approximation method for classical martingale differences instead of the characteristic function. In the rest of this section, we state some notations about the sublinear expectations. The main results on the central limit theorem and functional central limit theorem are stated in the next section. The proofs are given in Section 3.

At the last section, we consider general martingales and the Lévy characterization of a G-Brownian motion. The Lévy characterization of a G -Brownian motion is established by Xu and Zhang (2009, 2010) and extended by Lin (2013) by the method of the stochastic calculus. We will give an elementary proof without using stochastic calculus. We will find that the functional central limit theorem gives a new way to show the Lévy characterization.

We use the framework and notations of Peng (2008b). Let (Ω, \mathcal{F}) be a given measurable space and let \mathcal{H} be a linear space of real functions defined on (Ω, \mathcal{F}) such that if $X_1, \dots, X_n \in \mathcal{H}$ then $\varphi(X_1, \dots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}_n)$, where $C_{l,Lip}(\mathbb{R}_n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leq C(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)|\mathbf{x} - \mathbf{y}|, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}_n,$$

for some $C > 0, m \in \mathbb{N}$ depending on φ .

\mathcal{H} is considered as a space of “random variables”. In this case we denote $X \in \mathcal{H}$. We also denote the space of bounded Lipschitz functions and the space of bounded continuous functions on \mathbb{R}_n by $C_{b,Lip}(\mathbb{R}_n)$ and $C_b(\mathbb{R}_n)$, respectively.

Definition 1.1 *A sublinear expectation $\widehat{\mathbb{E}}$ on \mathcal{H} is a function $\widehat{\mathbb{E}} : \mathcal{H} \rightarrow \overline{\mathbb{R}}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$,*

- (a) *Monotonicity: If $X \geq Y$ then $\widehat{\mathbb{E}}[X] \geq \widehat{\mathbb{E}}[Y]$;*

(b) *Constant preserving:* $\widehat{\mathbb{E}}[c] = c$;

(c) *Sub-additivity:* $\widehat{\mathbb{E}}[X+Y] \leq \widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$ whenever $\widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;

(d) *Positive homogeneity:* $\widehat{\mathbb{E}}[\lambda X] = \lambda \widehat{\mathbb{E}}[X]$, $\lambda \geq 0$.

Here $\overline{\mathbb{R}} = [-\infty, \infty]$. The triple $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is called a sublinear expectation space. Give a sublinear expectation $\widehat{\mathbb{E}}$, let us denote the conjugate expectation $\widehat{\mathcal{E}}$ of $\widehat{\mathbb{E}}$ by $\widehat{\mathcal{E}}[X] := -\widehat{\mathbb{E}}[-X]$, $\forall X \in \mathcal{H}$.

If X is not in \mathcal{H} , we define its sublinear expectation by $\widehat{\mathbb{E}}^*[X] = \inf\{\widehat{\mathbb{E}}[Y] : X \leq Y \in \mathcal{H}\}$. When there is no ambiguity, we also denote it by $\widehat{\mathbb{E}}$. From the definition, it is easily shown that $\widehat{\mathcal{E}}[X] \leq \widehat{\mathbb{E}}[X]$, $\widehat{\mathbb{E}}[X+c] = \widehat{\mathbb{E}}[X] + c$ and $\widehat{\mathbb{E}}[X-Y] \geq \widehat{\mathbb{E}}[X] - \widehat{\mathbb{E}}[Y]$ for all $X, Y \in \mathcal{H}$ with $\widehat{\mathbb{E}}[Y]$ being finite. Further, if $\widehat{\mathbb{E}}[|X|]$ is finite, then $\widehat{\mathcal{E}}[X]$ and $\widehat{\mathbb{E}}[X]$ are both finite.

Definition 1.2 (Peng (2006, 2008b))

(i) (Identical distribution) Let \mathbf{X}_1 and \mathbf{X}_2 be two n -dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \widehat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \widehat{\mathbb{E}}_2)$. They are called identically distributed, denoted by $\mathbf{X}_1 \stackrel{d}{=} \mathbf{X}_2$ if

$$\widehat{\mathbb{E}}_1[\varphi(\mathbf{X}_1)] = \widehat{\mathbb{E}}_2[\varphi(\mathbf{X}_2)], \quad \forall \varphi \in C_{l,Lip}(\mathbb{R}_n),$$

whenever the sub-expectations are finite. A sequence $\{X_n; n \geq 1\}$ of random variables is said to be identically distributed if $X_i \stackrel{d}{=} X_1$ for each $i \geq 1$.

(ii) (Independence) In a sublinear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$, a random vector $\mathbf{Y} = (Y_1, \dots, Y_n)$, $Y_i \in \mathcal{H}$ is said to be independent to another random vector $\mathbf{X} = (X_1, \dots, X_m)$, $X_i \in \mathcal{H}$ under $\widehat{\mathbb{E}}$ if for each test function $\varphi \in C_{l,Lip}(\mathbb{R}_m \times \mathbb{R}_n)$ we have $\widehat{\mathbb{E}}[\varphi(\mathbf{X}, \mathbf{Y})] = \widehat{\mathbb{E}}[\widehat{\mathbb{E}}[\varphi(\mathbf{x}, \mathbf{Y})] |_{\mathbf{x}=\mathbf{X}}]$, whenever $\overline{\varphi}(\mathbf{x}) := \widehat{\mathbb{E}}[|\varphi(\mathbf{x}, \mathbf{Y})|] < \infty$ for all \mathbf{x} and $\widehat{\mathbb{E}}[|\overline{\varphi}(\mathbf{X})|] < \infty$.

Random variables X_1, \dots, X_n are said to be independent if for each $2 \leq k \leq n$, X_k is independent to (X_1, \dots, X_{k-1}) . A sequence of random variables is said to be independent if for each n , X_1, \dots, X_n are independent.

Next, we introduce the capacities corresponding to the sublinear expectations. Let $\mathcal{G} \subset \mathcal{F}$. A function $V : \mathcal{G} \rightarrow [0, 1]$ is called a capacity if

$$V(\emptyset) = 0, \quad V(\Omega) = 1 \quad \text{and} \quad V(A) \leq V(B) \quad \forall A \subset B, \quad A, B \in \mathcal{G}.$$

It is called to be sub-additive if $V(A \cup B) \leq V(A) + V(B)$ for all $A, B \in \mathcal{G}$ with $A \cup B \in \mathcal{G}$.

We denote a pair $(\mathbb{V}, \mathcal{V})$ of capacities on $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ by

$$\mathbb{V}(A) := \inf\{\widehat{\mathbb{E}}[\xi] : I_A \leq \xi, \xi \in \mathcal{H}\}, \quad \mathcal{V}(A) := 1 - \mathbb{V}(A^c), \quad \forall A \in \mathcal{F},$$

where A^c is the complement set of A . Then

$$\begin{aligned} \mathbb{V}(A) &:= \widehat{\mathbb{E}}[I_A], \quad \mathcal{V}(A) := \widehat{\mathcal{E}}[I_A], \quad \text{if } I_A \in \mathcal{H} \\ \widehat{\mathbb{E}}[f] &\leq \mathbb{V}(A) \leq \widehat{\mathbb{E}}[g], \quad \widehat{\mathcal{E}}[f] \leq \mathcal{V}(A) \leq \widehat{\mathcal{E}}[g], \quad \text{if } f \leq I_A \leq g, f, g \in \mathcal{H}. \end{aligned} \tag{1.1}$$

It is obvious that \mathbb{V} is sub-additive. But \mathcal{V} and $\widehat{\mathcal{E}}$ are not. However, we have

$$\mathcal{V}(A \cup B) \leq \mathcal{V}(A) + \mathbb{V}(B) \quad \text{and} \quad \widehat{\mathcal{E}}[X + Y] \leq \widehat{\mathcal{E}}[X] + \widehat{\mathbb{E}}[Y]$$

due to the fact that $\mathbb{V}(A^c \cap B^c) = \mathbb{V}(A^c \setminus B) \geq \mathbb{V}(A^c) - \mathbb{V}(B)$ and $\widehat{\mathbb{E}}[-X - Y] \geq \widehat{\mathbb{E}}[-X] - \widehat{\mathbb{E}}[Y]$.

We define the Choquet integrals/expecations of $(C_{\mathbb{V}}, C_{\mathcal{V}})$ by

$$C_V[X] = \int_0^\infty V(X \geq t) dt + \int_{-\infty}^0 [V(X \geq t) - 1] dt$$

with V being replaced by \mathbb{V} and \mathcal{V} respectively. It can be verified that (cf., Lemma 4.4 of Zhang (2016)), if $\lim_{c \rightarrow \infty} \widehat{\mathbb{E}}[|X| \wedge c] = \widehat{\mathbb{E}}[|X|]$ or $\widehat{\mathbb{E}}$ is countably sub-additive, i.e.,

$$\widehat{\mathbb{E}}\left[\sum_{i=1}^{\infty} X_i\right] \leq \sum_{i=1}^{\infty} \widehat{\mathbb{E}}[X_i], \quad \text{for all random variables } X_i \geq 0,$$

then $\widehat{\mathbb{E}}[|X|] \leq C_{\mathbb{V}}(|X|)$.

Finally, we give the notations of G-normal distribution and G-Brownian motion which are introduced by Peng (2008b, 2010).

Definition 1.3 (G-normal random variable) *For $0 \leq \underline{\sigma}^2 \leq \overline{\sigma}^2 < \infty$, a random variable ξ in a sub-linear expectation space $(\widetilde{\Omega}, \widetilde{\mathcal{H}}, \widetilde{\mathbb{E}})$ is called a normal $N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$ distributed (write $X \sim N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$ under $\widetilde{\mathbb{E}}$), if for any $\varphi \in C_{l,Lip}(\mathbb{R})$, the function $u(x, t) = \widetilde{\mathbb{E}}[\varphi(x + \sqrt{t}\xi)]$ ($x \in \mathbb{R}, t \geq 0$) is the unique viscosity solution of the following heat equation:*

$$\partial_t u - G(\partial_{xx}^2 u) = 0, \quad u(0, x) = \varphi(x),$$

where $G(\alpha) = \frac{1}{2}(\overline{\sigma}^2 \alpha^+ - \underline{\sigma}^2 \alpha^-)$.

That X is a normal distributed random variable is equivalent to that, if X' is an independent copy of X , then

$$\tilde{\mathbb{E}}[\varphi(\alpha X + \beta X')] = \tilde{\mathbb{E}}\left[\varphi(\sqrt{\alpha^2 + \beta^2} X)\right], \quad \forall \varphi \in C_{l,Lip}(\mathbb{R}) \text{ and } \forall \alpha, \beta \geq 0,$$

(c.f. Definition II.1.4 and Example II.1.13 of Peng (2010)).

Definition 1.4 (*G*-Browian motion) *A random process $(W_t)_{t \geq 0}$ in the sub-linear expectation space $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{\mathbb{E}})$ is called a *G*-Brownian motion (c.f. Definition III.1.2 of Peng (2010)) if*

(i) $W_0 = 0$;

(ii) For each $0 \leq t_1 \leq \dots \leq t_d \leq t \leq s$,

$$\begin{aligned} & \tilde{\mathbb{E}}[\varphi(W_{t_1}, \dots, W_{t_d}, W_s - W_t)] \\ &= \tilde{\mathbb{E}}\left[\tilde{\mathbb{E}}[\varphi(x_1, \dots, x_d, (t-s)\xi)] \mid_{x_1=W_{t_1}, \dots, x_d=W_{t_d}}\right] \\ & \quad \forall \varphi \in C_{l,Lip}(\mathbb{R}_{d+1}), \end{aligned} \tag{1.2}$$

where $\xi \sim N(0, [\underline{\sigma}^2, \bar{\sigma}^2])$.

In some papers, for example, Xu and Zhang (2009 2010), the test functions φ are only required to be elements in $C_{b,Lip}(\mathbb{R}_{d+1})$. It can be shown that if $\tilde{\mathbb{E}}[|W_t|^p] < \infty$ for all $p > 0$ and t , then that (1.2) holds for all $\varphi \in C_{b,Lip}(\mathbb{R}_{d+1})$ is equivalent to that it holds for all $\varphi \in C_{l,Lip}(\mathbb{R}_{d+1})$. Further, if the sub-linear expectation $\tilde{\mathbb{E}}$ is countably sub-additive, then this two kinds of definitions are equivalent. In fact, if $\xi \sim N(0, [\underline{\sigma}^2, \bar{\sigma}^2])$ under $\tilde{\mathbb{E}}$, then (c.f. Peng(2010, page 22))

$$\tilde{\mathbb{E}}[|\xi|^p] = \bar{\sigma}^p \int_{-\infty}^{\infty} |x|^p \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = c_p \bar{\sigma}^p, \quad \forall p \geq 1.$$

Suppose that X is a random variable in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ such that for any $\varphi \in C_{b,Lip}(\mathbb{R})$,

$$\hat{\mathbb{E}}[\varphi(X)] = \tilde{\mathbb{E}}[\varphi(\xi)], \quad \forall \varphi \in C_{b,Lip}(\mathbb{R}). \tag{1.3}$$

For any $z > 0$, one can choose a function $\varphi \in C_{b,Lip}(\mathbb{R})$ such that $I\{x > z\} \leq \varphi(x) \leq I\{x > z - \epsilon\}$. It follows that

$$\mathbb{V}(|X| > z) \leq \hat{\mathbb{E}}[\varphi(X)] = \tilde{\mathbb{E}}[\varphi(\xi)] \leq \tilde{\mathbb{V}}(|\xi| > z - \epsilon).$$

Hence

$$\mathbb{V}(|X| > z) \leq \tilde{\mathbb{V}}(|\xi| \geq z) \leq \frac{\tilde{\mathbb{E}}[|\xi|^{2p}]}{z^{2p}} = \frac{\bar{\sigma}^{2p} c_{2p}}{z^{2p}}.$$

It follows that

$$C_{\mathbb{V}}(|X|^p) = \int_0^\infty \mathbb{V}(|X|^p > z) dz \leq 1 + \int_1^\infty \frac{\bar{\sigma}^{2p} c_{2p}}{z^2} dz \leq 1 + \bar{\sigma}^{2p} c_{2p} < \infty, \quad \forall p \geq 2.$$

So, if $\widehat{\mathbb{E}}$ is countably sub-additive or $\widehat{\mathbb{E}}[|X|^p] = \lim_{c \rightarrow \infty} \widehat{\mathbb{E}}[(|X| \wedge c)^p]$, then $\widehat{\mathbb{E}}[|X|^p] \leq C_{\mathbb{V}}(|X|^p) < \infty$ for all $p > 0$.

2 Main results

We write $\eta_n \xrightarrow{\mathbb{V}} \eta$ if $\mathbb{V}(|\eta_n - \eta| \geq \epsilon) \rightarrow 0$ for any $\epsilon > 0$, and write $\eta_n \xrightarrow{d} \eta$ if $\widehat{\mathbb{E}}[\varphi(\eta_n)] \rightarrow \widehat{\mathbb{E}}[\varphi(\eta)]$ holds for any bounded and continuous function φ .

In this section, for simplifying the notations we consider the random variables which are functions of independent random variables. The general martingale-like random variables will be considered in the last section. Let $\{X_{n,k}; k = 1, \dots, k_n\}$ be an array of independent random variables. Let

$$\mathcal{H}_{n,k} = \{\varphi(X_{n,1}, \dots, X_{n,k}) : \varphi \in C_{l,Lip}(\mathbb{R}_k)\}$$

and denote $\mathcal{H}_{n,0}$ the space of constant random variables. For a random variable $Z = \varphi(X_{n,1}, \dots, X_{n,k_n})$, $\varphi \in C_{l,Lip}(\mathbb{R}_{k_n})$, we denote

$$\begin{aligned} \widehat{\mathbb{E}}[Z|\mathcal{H}_{n,k}] &= \widehat{\mathbb{E}}[Z|X_{n,1}, \dots, X_{n,k}] \\ &= \widehat{\mathbb{E}}[\varphi(x_1, \dots, x_k, X_{n,k+1}, \dots, X_{n,k_n})] \Big|_{x_1=X_{n,1}, \dots, x_k=X_{n,k}}, \end{aligned}$$

whenever the sub-linear expectations considered are finite. Also, for a X being a Borel function of $\{X_{n,k}; k = 1, \dots, k_n\}$, we define

$$\widehat{\mathbb{E}}^*[X|\mathcal{H}_{n,k}] = \inf \left\{ \widehat{\mathbb{E}}[Y|\mathcal{H}_{n,k}] : X \leq Y \in \mathcal{H}_{n,k_n} \text{ and } \widehat{\mathbb{E}}[Y|\mathcal{H}_{n,k}] \in \mathcal{H}_{n,k} \right\}.$$

Suppose that $\{Z_{n,k}; k = 1, \dots, k_n\}$ is an array of random variables such that $Z_{n,k} \in \mathcal{H}_{n,k}$, $\widehat{\mathbb{E}}[f|\mathcal{H}_{n,k-1}] \in \mathcal{H}_{n,k-1}$ for $f = Z_{n,k}, -Z_{n,k}, Z_{n,k}^2, -Z_{n,k}^2, (Z_{n,k}^2 - c)^+$.

Theorem 2.1 *Suppose that the following Lindeberg condition is satisfied:*

$$\sum_{k=1}^{k_n} \widehat{\mathbb{E}} \left[(Z_{n,k}^2 - \epsilon)^+ | \mathcal{H}_{n,k-1} \right] \xrightarrow{\mathbb{V}} 0 \quad \forall \epsilon > 0, \quad (2.1)$$

and further, there are constants $\rho \geq 0$ and $r \in [0, 1]$ such that

$$\sum_{k=1}^{k_n} \widehat{\mathbb{E}}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}] \xrightarrow{\mathbb{V}} \rho, \quad (2.2)$$

$$\sum_{k=1}^{k_n} \left| r \widehat{\mathbb{E}}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}] - \widehat{\mathcal{E}}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}] \right| \xrightarrow{\mathbb{V}} 0, \quad (2.3)$$

$$\sum_{k=1}^{k_n} \left\{ |\widehat{\mathbb{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}]| + |\widehat{\mathcal{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}]| \right\} \xrightarrow{\mathbb{V}} 0. \quad (2.4)$$

Then for any bounded continuous function φ ,

$$\lim_{n \rightarrow \infty} \widehat{\mathbb{E}} \left[\varphi \left(\sum_{k=1}^{k_n} Z_{n,k} \right) \right] = \widetilde{\mathbb{E}}[\varphi(\sqrt{\rho}\xi)], \quad (2.5)$$

i.e.,

$$\sum_{k=1}^{k_n} Z_{n,k} \xrightarrow{d} \sqrt{\rho}\xi,$$

where $\xi \sim N(0, [r, 1])$ under $\widetilde{\mathbb{E}}$.

Remark 2.1 When $\widehat{\mathbb{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}] = 0$ and $\widehat{\mathcal{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}] = 0$, then $\{Z_{n,k}; k = 1, \dots, k_n\}$ is an array of symmetric martingale differences (c.f. Xu and Zhang (2009)). If $\widehat{\mathbb{E}}[\cdot] = E_P[\cdot]$ is a classical linear expectation, then (2.3) is satisfied with $r = 1$, and the conclusion coincides with Corollary 3.1 of Hall and Heyde (1980). In Corollary 3.1 of Hall and Heyde (1980), ρ can be random. It is interesting to know whether the central limit theorem under the sublinear expectation holds or not when ρ is a random variable. We conjecture that if the conditions (2.1), (2.3), (2.4) are satisfied and

$$\sum_{k=1}^{k_n} \widehat{\mathbb{E}}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}] \xrightarrow{d} \rho, \text{ and } \rho \text{ is maximal distributed,}$$

i.e., there is an interval $[\underline{\mu}, \overline{\mu}]$ such that $\widehat{\mathbb{E}}[\varphi(\rho)] = \sup_{\underline{\mu} \leq x \leq \overline{\mu}} \varphi(x)$ for all $\varphi \in C_{l,ip}(\mathbb{R})$, then (2.5) holds with $\xi \sim N(0, [r, 1])$ being independent to ρ .

The following is a direct corollary of Theorem 2.1.

Corollary 2.1 Let $\{\eta_n\}$ be a sequence of independent random variables on $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ with $\widehat{\mathbb{E}}[\eta_n] = \widehat{\mathcal{E}}[\eta_n] = 0$, $\widehat{\mathbb{E}}[\eta_n^2] =: \overline{\sigma}_n^2 \rightarrow \overline{\sigma}^2$, $\widehat{\mathcal{E}}[\eta_n^2] =: \underline{\sigma}_n^2 \rightarrow \underline{\sigma}^2$ and $\sup_n \widehat{\mathbb{E}}[(\eta_n^2 - c)] \rightarrow 0$ as $c \rightarrow \infty$.

Suppose that $\{a_{n,i}; i = 1, \dots, k_n\}$ is an array of real random variables in \mathcal{H} with $a_{n,i}$ being a function of $\eta_1, \dots, \eta_{i-1}$,

$$\max_i |a_{n,i}| \xrightarrow{\mathbb{V}} 0 \text{ and } \sum_{i=1}^{k_n} a_{n,i}^2 \xrightarrow{\mathbb{V}} \rho,$$

where $\rho \geq 0$ is a constant. Then

$$\lim_{n \rightarrow \infty} \widehat{\mathbb{E}} \left[\varphi \left(\sum_{i=1}^{k_n} a_{n,i} \eta_i \right) \right] = \widetilde{\mathbb{E}}[\varphi(\xi)], \quad (2.6)$$

for any bounded continuous function φ , where $\xi \sim N(0, [\rho \underline{\sigma}^2, \rho \overline{\sigma}^2])$ under $\widetilde{\mathbb{E}}$.

The following corollary is a central limit theorem for moving average processes which include the ARMA model.

Corollary 2.2 *Let $\{\eta_n\}$ be a sequence of independent and identically distributed random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ with $\widehat{\mathbb{E}}[\eta_1] = \widehat{\mathcal{E}}[\eta_1] = 0$, $\widehat{\mathbb{E}}[\eta_1^2] = \overline{\sigma}^2$ and $\widehat{\mathcal{E}}[\eta_1^2] = \underline{\sigma}^2$, $\{a_n; n \geq 0\}$ be a sequence of real numbers with $\sum_{n=0}^{\infty} |a_n| < \infty$. Let $X_k = \sum_{i=0}^{\infty} a_i \eta_{i+k}$. Then*

$$\frac{1}{\sqrt{n}} \sum_{k=1}^n X_k \xrightarrow{d} N(0, [a^2 \underline{\sigma}^2, a^2 \overline{\sigma}^2]), \quad (2.7)$$

where $a = \sum_{j=0}^{\infty} a_j$.

Proof. Let $a_n = 0$ if $n < 0$. Then $X_k = \sum_{i=1}^{\infty} a_{i-k} \eta_i$ and

$$\frac{1}{\sqrt{n}} \sum_{k=1}^n X_k = \sum_{i=1}^{\infty} \left(\frac{\sum_{k=1}^n a_{i-k}}{\sqrt{n}} \right) \eta_i.$$

Let $a_{n,i} = \frac{\sum_{k=1}^n a_{i-k}}{\sqrt{n}}$. Then $\max_i |a_{n,i}| \leq n^{-1/2} \sum_{i=-\infty}^{\infty} |a_i| \rightarrow 0$ and $\sum_{i=1}^{\infty} a_{n,i}^2 \rightarrow a^2$. The result follows from Corollary 2.1. \square

If we consider the independent random variables $\{X_{n,k}; k = 1, \dots, k_n\}$, we have the following Lindeberg's central limit theorem. Denote $\overline{\sigma}_{n,k}^2 = \widehat{\mathbb{E}}[X_{n,k}^2]$, $\underline{\sigma}_{n,k}^2 = \widehat{\mathcal{E}}[X_{n,k}^2]$, $B_n^2 = \sum_{k=1}^{k_n} \overline{\sigma}_{n,k}^2$.

Theorem 2.2 *Suppose that the Lindeberg condition is satisfied:*

$$\frac{1}{B_n^2} \sum_{k=1}^{k_n} \widehat{\mathbb{E}} \left[(X_{n,k}^2 - \epsilon B_n^2)^+ \right] \rightarrow 0 \quad \forall \epsilon > 0, \quad (2.8)$$

and further, there is a constant $r \in [0, 1]$ such that

$$\frac{\sum_{k=1}^{k_n} |r\bar{\sigma}_{n,k}^2 - \underline{\sigma}_{n,k}^2|}{B_n^2} \rightarrow 0, \quad \text{also,} \quad (2.9)$$

$$\frac{\sum_{k=1}^{k_n} \left\{ |\widehat{\mathbb{E}}[X_{n,k}]| + |\widehat{\mathcal{E}}[X_{n,k}]| \right\}}{B_n} \rightarrow 0. \quad (2.10)$$

Then for any continuous function φ satisfying $|\varphi(x)| \leq C(1 + x^2)$,

$$\lim_{n \rightarrow \infty} \widehat{\mathbb{E}} \left[\varphi \left(\frac{\sum_{k=1}^{k_n} X_{n,k}}{B_n} \right) \right] = \widetilde{\mathbb{E}}[\varphi(\xi)], \quad (2.11)$$

where $\xi \sim N(0, [r, 1])$ under $\widetilde{\mathbb{E}}$. Further, if (2.8) is replaced by the condition that for some $p > 2$,

$$\frac{1}{B_n^p} \sum_{k=1}^{k_n} \widehat{\mathbb{E}} [|X_{n,k}|^p] \rightarrow 0, \quad (2.12)$$

then (2.11) holds for any continuous function φ satisfying $|\varphi(x)| \leq C(1 + |x|^p)$.

Remark 2.2 Li and Shi (2010) established a central limit theorem for independent random variables $\{X_n; n \geq 1\}$ satisfying $\widehat{\mathbb{E}}[X_i] = \widehat{\mathcal{E}}[X_i] = 0$, $\widehat{\mathbb{E}}[|X_i|^3] \leq M < \infty$, $i = 1, 2, \dots$, and

$$\frac{1}{n} \sum_{i=1}^n \left| \widehat{\mathbb{E}}[X_i^2] - \bar{\sigma}^2 \right| \rightarrow 0, \quad \frac{1}{n} \sum_{i=1}^n \left| \widehat{\mathcal{E}}[X_i^2] - \underline{\sigma}^2 \right| \rightarrow 0.$$

It is easily seen that the array $\{\frac{1}{\sqrt{n}}X_k; k = 1, \dots, n\}$ satisfies the conditions (2.9), (2.10) and (2.12) with $p = 3$ and $r = \underline{\sigma}^2/\bar{\sigma}^2$.

Remark 2.3 It is easily seen that (2.9) implies

$$\frac{\sum_{k=1}^{k_n} \underline{\sigma}_{n,k}^2}{\sum_{k=1}^{k_n} \bar{\sigma}_{n,k}^2} \rightarrow r. \quad (2.13)$$

One may conjecture that (2.9) can be weakened to (2.13). The following example tells us that it is not the truth.

Example 2.1 Let $0 < \tau < 1$, and $\{X_{n,k}; k = 1, \dots, 2n\}$ be a sequence of independent normal random variables such that

$$X_{n,k} \sim N(0, [\tau, 1]), k = 1, \dots, n \text{ and } X_{n,k} \sim aN(0, [1, 1]), k = n + 1, \dots, 2n.$$

It is easily seen that $\{X_{n,k}; k = 1, \dots, 2n\}$ satisfies the conditions (2.8), (2.10) and (2.13) with $r = (\tau + a^2)/(1 + a^2)$, and $B_n^2 = (1 + a^2)n$. It is obvious that

$$\frac{\sum_{k=1}^{2n} X_{n,k}}{\sqrt{n}} = \frac{\sum_{k=1}^n X_{n,k}}{\sqrt{n}} + \frac{\sum_{k=n+1}^{2n} X_{n,k}}{\sqrt{n}} \sim \xi + a\eta,$$

where ξ, η are independent normal random variables with $\xi \stackrel{d}{\sim} N(0, [\tau, 1])$, $\eta \sim N(0, [1, 1])$.

We can show that for $|a| \geq 6$, $\xi + a\eta$ is not G -normal distributed, and hence (2.11) fails.

Proof. By noting that x^+ is a convex function, it follows that (c.f. Peng(2010, page 22))

$$\widehat{\mathbb{E}}[\xi^+] = \int_{-\infty}^{\infty} x^+ \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}}.$$

Let ξ_1 be a random variable which is independent to ξ with $\xi_1 \stackrel{d}{=} \xi$. Then

$$\begin{aligned} \widehat{\mathbb{E}}[(\xi + \xi_1)^3 - \xi^3 - \xi_1^3] &= \widehat{\mathbb{E}}[3\xi^2\xi_1 + 3\xi\xi_1^2] = 3\widehat{\mathbb{E}}[\xi\xi_1^2] \\ &= 3\widehat{\mathbb{E}}[\xi^+ - \xi^-\tau] = 3\widehat{\mathbb{E}}[\xi^+(1 - \tau) + \xi\tau] = 3(1 - \tau)\widehat{\mathbb{E}}[\xi^+] = \frac{3}{\sqrt{2\pi}}(1 - \tau). \end{aligned}$$

On the other hand,

$$\widehat{\mathbb{E}}[(\xi + \xi_1)^3 - \xi^3 - \xi_1^3] \leq \widehat{\mathbb{E}}[(\xi + \xi_1)^3] + \widehat{\mathbb{E}}[(-\xi)^3] + \widehat{\mathbb{E}}[(-\xi_1)^3] = (2\sqrt{2} + 2)\widehat{\mathbb{E}}[\xi^3],$$

$$\widehat{\mathbb{E}}[(\xi + \xi_1)^3 - \xi^3 - \xi_1^3] \geq \widehat{\mathbb{E}}[(\xi + \xi_1)^3] - \widehat{\mathbb{E}}[\xi^3 + \xi_1^3] = (2\sqrt{2} - 2)\widehat{\mathbb{E}}[\xi^3].$$

We conclude that

$$\frac{3(2 - \sqrt{2})}{4\sqrt{\pi}}(1 - \tau) \leq \widehat{\mathbb{E}}[\xi^3] \leq \frac{3(2 + \sqrt{2})}{4\sqrt{\pi}}(1 - \tau). \quad (2.14)$$

Now

$$\begin{aligned} \widehat{\mathbb{E}}[(\xi + a\eta)^3] &= \widehat{\mathbb{E}}\left[\widehat{\mathbb{E}}\left[x^3 + 3x^2a\eta + 3xa^2\eta^2 + a^3\eta^3 \middle| x=\xi\right]\right] \\ &= \widehat{\mathbb{E}}[\xi^3 + 3\xi a^2] = \widehat{\mathbb{E}}[\xi^3] \leq \frac{3(2 + \sqrt{2})}{4\sqrt{\pi}}(1 - \tau) \end{aligned} \quad (2.15)$$

by (2.14). If $\xi + a\eta$ is G -normal distributed, then $\xi + a\eta \sim \sqrt{1 + a^2}N(0, [r, 1])$ where $r = (\tau + a^2)/(1 + a^2)$. Then by (2.14) again,

$$\widehat{\mathbb{E}}[(\xi + a\eta)^3] \geq (1 + a^2)^{3/2} \frac{3(2 - \sqrt{2})}{4\sqrt{\pi}}(1 - r) = (1 + a^2)^{1/2} \frac{3(2 - \sqrt{2})}{4\sqrt{\pi}}(1 - \tau),$$

which contradicts (2.15) when $|a| \geq 6$, and so $\xi + a\eta$ is impossibly G -normal distributed. \square

Finally, we give the functional central limit theorems. We first consider the independent random variables $\{X_{n,k}; k = 1, \dots, k_n\}$. Let $S_{n,k} = \sum_{i=1}^k X_{n,i}$, $k \leq k_n$. Define a random function as

$$W_n(t) = \begin{cases} S_{n,k}/B_n, & \text{if } t = \sum_{i=1}^k \bar{\sigma}_{n,i}^2/B_n^2, \\ \text{extended by linear interpolation in each interval} \\ [\sum_{i=1}^{k-1} \bar{\sigma}_{n,i}^2/B_n^2, \sum_{i=1}^k \bar{\sigma}_{n,i}^2/B_n^2], \end{cases}$$

where $\sum_{k=1}^0 [\cdot] = 0$.

Theorem 2.3 *Suppose that the conditions (2.8), (2.9) and (2.10) are satisfied. Then for any continuous function $\varphi : C_{[0,1]} \rightarrow \mathbb{R}$ satisfying $|\varphi(x)| \leq C(1 + \|x\|^2)$,*

$$\lim_{n \rightarrow \infty} \widehat{\mathbb{E}}[\varphi(W_n)] = \widetilde{\mathbb{E}}[\varphi(W)], \quad (2.16)$$

where W is G -Browian motion on $[0, 1]$ with $W(1) \sim N(0, [r, 1])$ under $\widetilde{\mathbb{E}}$. Further, if (2.8) is replaced by (2.12) for some $p > 2$, then (2.16) holds for any continuous function φ satisfying $|\varphi(x)| \leq C(1 + \|x\|^p)$. Here $C_{[0,1]}$ is the space of continuous functions $x(t) : [0, 1] \rightarrow \mathbb{R}$, and $\|x\| = \sup_t |x(t)|$.

The following theorem is on the martingale-like random variables.

Theorem 2.4 *Suppose that the conditions (2.1), (2.3) and (2.4) in Theorem 2.1 are satisfied. Further, there is a continuous non-decreasing non-random function $\rho(t)$ such that*

$$\sum_{k \leq k_n t} \widehat{\mathbb{E}}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}] \xrightarrow{\mathbb{V}} \rho(t), \quad t \in [0, 1]. \quad (2.17)$$

Let $S_{n,i} = \sum_{k=1}^i Z_{n,k}$,

$$W_n(t) = \begin{cases} S_{n,k}, & \text{if } t = k/k_n, \\ \text{extended by linear interpolation in each interval} \\ [k/k_n, (k+1)/k_n]. \end{cases} \quad (2.18)$$

Then for any bounded continuous function $\varphi : C_{[0,1]} \rightarrow \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \widehat{\mathbb{E}}[\varphi(W_n)] = \widetilde{\mathbb{E}}[\varphi(W \circ \rho)], \quad (2.19)$$

where W is G -Browian motion on $[0, 1]$ with $W(1) \sim N(0, [r, 1])$ under $\widetilde{\mathbb{E}}$, and $W \circ \rho(t) = W(\rho(t))$.

3 Proofs

To prove the theorems, we need some lemma. The first is Hölder's inequality which is Proposition 16 of Denis, Hu and Peng (2011).

Lemma 3.1 (Hölder's inequality) *Let $p, q > 1$ be two real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then for two random variables X, Y in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ we have*

$$\hat{\mathbb{E}}[|XY|] \leq \left(\hat{\mathbb{E}}[|X|^p] \right)^{\frac{1}{p}} \left(\hat{\mathbb{E}}[|Y|^q] \right)^{\frac{1}{q}}$$

whenever $\hat{\mathbb{E}}[|X|^p] < \infty$, $\hat{\mathbb{E}}[|Y|^q] < \infty$.

For the martingale-difference like random variables, we have the following lemma on the Rosenthal type inequalities.

Lemma 3.2 *Set $S_0 = 0$, $S_k = \sum_{i=1}^k Z_{n,i}$. Then*

$$\hat{\mathbb{E}} \left[\left(\max_{k \leq k_n} (S_{k_n} - S_k) \right)^2 \right] \leq \hat{\mathbb{E}} \left[\sum_{k=1}^{k_n} \hat{\mathbb{E}}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}] \right] \quad (3.1)$$

when $\hat{\mathbb{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}] \leq 0$, $k = 1, \dots, k_n$, and in general

$$\begin{aligned} \hat{\mathbb{E}} \left[\max_{k \leq k_n} |S_k|^2 \right] &\leq 256 \left\{ \hat{\mathbb{E}} \left[\sum_{k=1}^{k_n} \hat{\mathbb{E}}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}] \right] \right. \\ &\quad \left. + \hat{\mathbb{E}} \left[\left\{ \sum_{k=1}^{k_n} \left((\hat{\mathbb{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}])^+ + (\hat{\mathbb{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}])^- \right) \right\}^2 \right] \right\}. \end{aligned} \quad (3.2)$$

Moreover, for $p \geq 2$ there is a constant C_p such that

$$\begin{aligned} \hat{\mathbb{E}} \left[\max_{k \leq k_n} |S_k|^p \right] &\leq C_p \left\{ \hat{\mathbb{E}} \left[\sum_{k=1}^{k_n} \hat{\mathbb{E}}[|Z_{n,k}|^p | \mathcal{H}_{n,k-1}] \right] + \hat{\mathbb{E}} \left[\left(\hat{\mathbb{E}} \left[\sum_{k=1}^{k_n} \hat{\mathbb{E}}[Z_{n,k}^2 | \mathcal{H}_{n,k}] \right] \right)^{p/2} \right] \right. \\ &\quad \left. + \hat{\mathbb{E}} \left[\left\{ \sum_{k=1}^{k_n} \left((\hat{\mathbb{E}}[Z_{n,k} | \mathcal{H}_{n,k}])^+ + (\hat{\mathbb{E}}[Z_{n,k} | \mathcal{H}_{n,k}])^- \right) \right\}^p \right] \right\}. \end{aligned} \quad (3.3)$$

Proof. Let $Q_k = \max\{Z_{n,k}, Z_{n,k} + Z_{n,k-1}, \dots, Z_{n,k} + \dots Z_{n,1}\}$, $M_k = \max_{i \leq k} |S_i|$. Then $Q_k = Z_{n,k} + Q_{k-1}^+$, $Q_k^2 = Z_{n,k}^2 + 2Z_{n,k}Q_{k-1}^+ + (Q_{k-1}^+)^2$, $|Q_k| \leq 2M_{k_n}$. It follows that

$$\begin{aligned}
& \left(\max_{k \leq k_n} (S_{k_n} - S_k) \right)^2 = (Q_{k_n}^+)^2 \leq \sum_{k=1}^{k_n} Z_{n,k}^2 + 2 \sum_{k=1}^{k_n} Z_{n,k} Q_{k-1}^+ \\
& \leq \sum_{k=1}^{k_n} \widehat{\mathbb{E}}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}] + \sum_{k=1}^{k_n} (Z_{n,k}^2 - \widehat{\mathbb{E}}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}]) \\
& \quad + 2 \sum_{k=1}^{k_n} \widehat{\mathbb{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}] Q_{k-1}^+ + 2 \sum_{k=1}^{k_n} (Z_{n,k} - \widehat{\mathbb{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}]) Q_{k-1}^+ \\
& \leq \sum_{k=1}^{k_n} \widehat{\mathbb{E}}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}] + 4 \sum_{k=1}^{k_n} (\widehat{\mathbb{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}])^+ M_{k_n} \\
& \quad + \sum_{k=1}^{k_n} (Z_{n,k}^2 - \widehat{\mathbb{E}}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}]) + 2 \sum_{k=1}^{k_n} (Z_{n,k} - \widehat{\mathbb{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}]) Q_{k-1}^+,
\end{aligned}$$

in which the sublinear expectation of the last two sums ≤ 0 , and the sublinear expectation of the second sum is also zero when $\widehat{\mathbb{E}}[Z_{n,k} | \mathcal{H}_{n,k}] \leq 0$, $k = 1, \dots, k_n$. Taking the the sublinear expectation yields (3.1). By considering $\{-Z_{n,k}\}$, for $\max_{k \leq k_n} (-S_{k_n} + S_k)$ we have a similar estimate. Note $M_{k_n} \leq 2 \max_{k \leq k_n} |S_n - S_k|$. It follows that

$$\begin{aligned}
\widehat{\mathbb{E}}[M_{k_n}^2] & \leq 8 \widehat{\mathbb{E}} \left[\sum_{k=1}^{k_n} \widehat{\mathbb{E}}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}] \right] \\
& \quad + 16 \widehat{\mathbb{E}} \left[\sum_{k=1}^{k_n} \{ (\widehat{\mathbb{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}])^+ + (\widehat{\mathcal{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}])^- \} M_{k_n} \right] \\
& \leq 8 \widehat{\mathbb{E}} \left[\sum_{k=1}^{k_n} \widehat{\mathbb{E}}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}] \right] + \frac{1}{2} \widehat{\mathbb{E}}[M_{k_n}^2] \\
& \quad + 128 \widehat{\mathbb{E}} \left[\left(\sum_{k=1}^{k_n} \{ (\widehat{\mathbb{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}])^+ + (\widehat{\mathcal{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}])^- \} \right)^2 \right],
\end{aligned}$$

where the last inequality is due to $ab \leq \frac{a^2+b^2}{2}$.

For (3.3), we apply the elementary inequality

$$|x + y|^p \leq 2^p p^2 |x|^p + |y|^p + p x |y|^{p-1} \text{sgn} y + 2^p p^2 x^2 |y|^{p-2}, \quad p \geq 2,$$

and yields

$$|Q_k|^p \leq 2^p p^2 |X_{n,k}|^p + |Q_{k-1}|^p + p X_{n,k} (Q_{k-1}^+)^{p-1} + 2^p p^2 X_{n,k}^2 (Q_{k-1}^+)^{p-2}.$$

It follows that

$$\begin{aligned}
& \left(\max_{k \leq k_n} (S_{k_n} - S_k) \right)^p \leq |Q_{k_n}|^p \\
& \leq 2^p p^2 \sum_{k=1}^{k_n} |Z_{n,k}|^p + p \sum_{k=1}^{k_n} Z_{n,k} (Q_{k-1}^+)^{p-1} + 2^p p^2 \sum_{k=1}^{k_n} Z_{n,k}^2 (Q_{k-1}^+)^{p-2} \\
& \leq 2^p p^2 \sum_{k=1}^{k_n} \widehat{\mathbb{E}}[|Z_{n,k}|^p | \mathcal{H}_{n,k-1}] + p \sum_{k=1}^{k_n} (\widehat{\mathbb{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}])^+ (Q_{k-1}^+)^{p-1} \\
& \quad + 2^p p^2 \sum_{k=1}^{k_n} \widehat{\mathbb{E}}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}] (Q_{k-1}^+)^{p-2} \\
& \quad + 2^p p^2 \sum_{k=1}^{k_n} \left(|Z_{n,k}|^p - \widehat{\mathbb{E}}[|Z_{n,k}|^p | \mathcal{H}_{n,k-1}] \right) \\
& \quad + p \sum_{k=1}^{k_n} (Z_{n,k} - \widehat{\mathbb{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}]) (Q_{k-1}^+)^{p-1} \\
& \quad + 2^p p^2 \sum_{k=1}^{k_n} (Z_{n,k}^2 - \widehat{\mathbb{E}}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}]) (Q_{k-1}^+)^{p-2}.
\end{aligned}$$

The sublinear expectations of the last three sums ≤ 0 . Note $Q_k \leq 2M_{k_n}$ and for $((\max_{k \leq k_n} (-S_{k_n} + S_k))^p)$ we have a similar estimate. It follows that

$$\begin{aligned}
\widehat{\mathbb{E}}[M_{k_n}^p] & \leq C_p \left\{ \widehat{\mathbb{E}} \left[\sum_{k=1}^{k_n} \widehat{\mathbb{E}}[|Z_{n,k}|^p | \mathcal{H}_{n,k-1}] \right] + \widehat{\mathbb{E}} \left[\sum_{k=1}^{k_n} \widehat{\mathbb{E}}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}] M_{k_n}^{p-2} \right] \right. \\
& \quad \left. + \widehat{\mathbb{E}} \left[\sum_{k=1}^{k_n} \left\{ (\widehat{\mathbb{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}])^+ + (\widehat{\mathbb{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}])^- \right\} M_{k_n}^{p-1} \right] \right\} \\
& \leq C_p \left\{ \widehat{\mathbb{E}} \left[\sum_{k=1}^{k_n} \widehat{\mathbb{E}}[|Z_{n,k}|^p | \mathcal{H}_{n,k-1}] \right] + \widehat{\mathbb{E}} \left[\left(\sum_{k=1}^{k_n} \widehat{\mathbb{E}}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}] \right)^{p/2} \right] \right. \\
& \quad \left. + \widehat{\mathbb{E}} \left[\left(\sum_{k=1}^{k_n} \left\{ (\widehat{\mathbb{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}])^+ + (\widehat{\mathbb{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}])^- \right\} \right)^p \right] \right\} + \frac{1}{2} \widehat{\mathbb{E}}[M_{k_n}^p],
\end{aligned}$$

where the last inequality is due to $ab \leq \frac{2}{p}|a|^{p/2} + (1 - \frac{2}{p})|b|^{p/(p-2)}$ and $ab \leq \frac{1}{p}|a|^p + (1 - \frac{1}{p})|b|^{p/(p-1)}$. The proof is completed. \square

The following Rosenthal type inequality for independent random variables obtained by Zhang (2015a) is a special case of (3.3).

Lemma 3.3 *Let $\{X_1, \dots, X_n\}$ be a sequence of independent random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$.*

Then

$$\begin{aligned} \widehat{\mathbb{E}} \left[\max_{k \leq n} |S_k|^p \right] \leq C_p \left\{ \sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p] + \left(\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^2] \right)^{p/2} \right. \\ \left. + \left(\sum_{k=1}^n [(\widehat{\mathcal{E}}[X_k])^- + (\widehat{\mathbb{E}}[X_k])^+] \right)^p \right\}, \quad \text{for } p \geq 2. \end{aligned} \quad (3.4)$$

Proof of Theorem 2.1. Denote $S_0 = 0$, $\delta_0 = 0$, $S_k = \sum_{i=1}^k Z_{n,i}$, $a_{n,k}^2 = \widehat{\mathbb{E}}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}]$, $\delta_k = \sum_{i=1}^k a_{n,i}^2$, $k = 1, \dots, k_n$. By the Lindeberg condition (2.1) it follows that

$$\max_{i \leq k_n} |a_{n,i}| \xrightarrow{\mathbb{V}} 0.$$

Let $f(x)$ be a function with bounded derivative such that $I\{x \leq \rho + \epsilon/2\} \leq f(x) \leq I\{x \leq \rho + \epsilon\}$. Let $Z_{n,k}^* = Z_{n,k} f(\delta_k)$. Then $\{Z_{n,k}^*; k = 1, \dots, k_n\}$ satisfy the conditions (2.2)-(2.4), and

$$\delta_{k_n}^* = \sum_{k=1}^{k_n} \widehat{\mathbb{E}}[(Z_{n,k}^*)^2 | \mathcal{H}_{n,k-1}] \leq \rho + \epsilon$$

and

$$\left\{ \sum_{k=1}^i Z_{n,k} \neq \sum_{k=1}^i Z_{n,k}^* \text{ for some } i \right\} \subset \left\{ \sum_{k=1}^{k_n} a_{n,k}^2 > \rho + \epsilon/2 \right\}.$$

It follows that for any bounded function φ ,

$$\widehat{\mathbb{E}} \left[\left| \varphi \left(\sum_{k=1}^{k_n} Z_{n,k} \right) - \varphi \left(\sum_{k=1}^{k_n} Z_{n,k}^* \right) \right| \right] \leq 2 \sup_x |\varphi(x)| \mathbb{V} \left(\sum_{k=1}^{k_n} \widehat{\mathbb{E}}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}] > \rho + \epsilon/2 \right) \rightarrow 0.$$

So, without loss of generality we can assume that $\delta_{k_n} = \sum_{k=1}^{k_n} \widehat{\mathbb{E}}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}] \leq \rho + \epsilon$. Similarly, we can assume that $\chi_{k_n} =: \sum_{k=1}^{k_n} \left\{ |\widehat{\mathbb{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}]| + |\widehat{\mathcal{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}]| \right\} < \epsilon < 1$.

By Lemma 3.2,

$$\widehat{\mathbb{E}} \left[\max_{k \leq k_n} \left(\sum_{i=1}^k Z_{n,i} \right)^2 \right] \leq 256 \widehat{\mathbb{E}}[\delta_{k_n}] + 256 \widehat{\mathbb{E}}[\chi_{k_n}^2]. \quad (3.5)$$

If $\rho = 0$, then $\delta_{k_n} \xrightarrow{\mathbb{V}} 0$. Note $\chi_{k_n} \xrightarrow{\mathbb{V}} 0$. So $\widehat{\mathbb{E}} \left[\left(\sum_{i=1}^{k_n} Z_{n,i} \right)^2 \right] \rightarrow 0$, and then the result is obvious. When $\rho \neq 0$, without loss of generality we assume $\rho = 1$. Let φ be a bounded continuous function with bounded derivation. Without loss of generality, we assume $|\varphi(x)| \leq 1$. We want to show that

$$\widehat{\mathbb{E}}[\varphi(S_{k_n})] \rightarrow \widetilde{\mathbb{E}}[\varphi(\xi)]. \quad (3.6)$$

In the classical probability space, the above convergence is usually shown by verifying the convergence of the related characteristic functions (c.f. Hall and Heyde (1980), p. 60-63;

Pollard (1984), p. 171-174). As shown by Hu and Li (2014), the characteristic function cannot determine the distribution of random variables in the sub-linear expectation space. Peng (2006, 2008b) developed a wonderful method to show the above convergence for independent random variables. Here we promote Peng's argument such that it is also valid for martingale differences which give also a new normal approximation method for classical martingale differences instead of the characteristic function.

Now, for a small but fixed $h > 0$, let $V(t, x)$ be the unique viscosity solution of the following equation,

$$\partial_t V + G(\partial_{xx}^2 V) = 0, \quad (t, x) \in [0, 1+h] \times \mathbb{R}, \quad V|_{t=1+h} = \varphi(x), \quad (3.7)$$

where $G(\alpha) = \frac{1}{2}(\alpha^+ - r\alpha^-)$. Then by the interior regularity of V ,

$$\|V\|_{C^{1+\alpha/2, 2+\alpha}([0, 1+h/2] \times \mathbb{R})} < \infty, \quad \text{for some } \alpha \in (0, 1). \quad (3.8)$$

According to the definition of G -normal distribution, we have $V(t, x) = \tilde{\mathbb{E}}[\varphi(x + \sqrt{1+h-t}\xi)]$ where $\xi \sim N(0, [r, 1])$ under $\tilde{\mathbb{E}}$. In particular,

$$V(h, 0) = \tilde{\mathbb{E}}[\varphi(\xi)], \quad V(1+h, x) = \varphi(x).$$

It is obvious that, if $\varphi(\cdot)$ is a global Lipschitz function, i.e., $|\varphi(x) - \varphi(y)| \leq C|x - y|$, then $|V(t, x) - V(t, y)| \leq C|x - y|$ and $|V(t, x) - V(s, x)| \leq C\tilde{\mathbb{E}}[|\xi|]|t - s|^{1/2}$. So, $|\partial_x V(t, x)| \leq C$, $|V(1+h, x) - V(1, x)| \leq C\tilde{\mathbb{E}}[|\xi|]\sqrt{h}$ and $|V(h, 0) - V(0, 0)| \leq C\tilde{\mathbb{E}}[|\xi|]\sqrt{h}$. Following the proof of Lemma 5.4 of Peng (2008b), it is sufficient to show that

$$\lim_{n \rightarrow \infty} \widehat{\mathbb{E}}[V(1, S_{k_n})] = V(0, 0). \quad (3.9)$$

As we have shown, we can assume that $\delta_{k_n} \leq \rho + h/4 < 2$. It is obvious that $|V(t, x)| \leq 1$, and

$$\widehat{\mathbb{E}} \left[\left| V(1, S_{k_n}) - V(\delta_{k_n}, S_{k_n}) \right| \right] \leq C\widehat{\mathbb{E}} \left[|\delta_{k_n} - 1|^{1/2} \right] \rightarrow 0.$$

Hence, it is sufficient to show that

$$\lim_{n \rightarrow \infty} \widehat{\mathbb{E}}[V(\delta_{k_n}, S_{k_n})] = V(0, 0). \quad (3.10)$$

Also, if let $f(x)$ be a smooth function such that $I\{x \leq N\} \leq f(x) \leq I\{x \leq 2N\}$, $Z_{n,k}^* = Z_{n,k} f(\max_{i \leq k-1} |S_i|)$, $\delta_k^* = \sum_{i=1}^k \widehat{\mathbb{E}}[(Z_{n,i}^*)^2 | \mathcal{H}_{n,i-1}] = \sum_{i=1}^k f^2(\max_{j \leq i-1} |S_j|) \widehat{\mathbb{E}}[Z_{n,i}^2 | \mathcal{H}_{n,i-1}]$

and $S_k^* = \sum_{i=1}^k Z_{n,i}^*$, then, by (3.5),

$$\begin{aligned} \sup_n \widehat{\mathbb{E}} [|V(\delta_{k_n}, S_{k_n}) - V(\delta_{k_n}^*, S_{k_n}^*)|] &\leq 2 \sup_n \mathbb{P} \left(\max_{i \leq k_n} |S_i| > N \right) \\ &\leq 2N^{-2} \widehat{\mathbb{E}} \left[\max_{i \leq k_n} |S_i|^2 \right] \leq CN^{-2} \sup_n \left\{ \widehat{\mathbb{E}}[\delta_{k_n}] + \widehat{\mathbb{E}}[\chi_{k_n}^2] \right\} \leq CN^{-2} \rightarrow 0 \end{aligned}$$

as $N \rightarrow \infty$. Hence it is sufficient to show that $\widehat{\mathbb{E}} [V(\delta_{k_n}^*, S_{k_n}^*)] \rightarrow V(0, 0)$, and then without loss of generality, we assume that $\max_{i \leq k-1} |S_i| \leq 2N$ whenever $\widehat{\mathbb{E}}[Z_{n,i}^2 | \mathcal{H}_{n,i-1}] \neq 0$, or $Z_{n,i} \neq 0$ or $\widehat{\mathbb{E}}[Z_{n,i} | \mathcal{H}_{n,i-1}] \neq 0$. Applying the Taylor's expansion yields

$$\begin{aligned} &V(\delta_{k_n}, S_{k_n}) - V(0, 0) \\ &= \sum_{i=0}^{k_n-1} \{ [V(\delta_{i+1}, S_{i+1}) - V(\delta_i, S_{i+1})] + [V(\delta_i, S_{i+1}) - V(\delta_i, S_i)] \} \\ &= \sum_{i=0}^{k_n-1} \{ I_n^i + J_n^i \}, \end{aligned}$$

with

$$\begin{aligned} J_n^i &= \partial_t V(\delta_i, S_i) a_{n,i+1}^2 + \frac{1}{2} \partial_{xx}^2 V(\delta_i, S_i) Z_{n,i+1}^2 + \partial_x V(\delta_i, S_i) Z_{n,i+1} \\ &= \left[a_{n,i+1}^2 \partial_t V(\delta_i, S_i) + \frac{1}{2} \partial_{xx}^2 V(\delta_i, S_i) Z_{n,i+1}^2 - \frac{1}{2} (\partial_{xx}^2 V(\delta_i, S_i))^- (ra_{n,i+1}^2 - \widehat{\mathcal{E}}[Z_{n,i+1}^2 | \mathcal{H}_{n,i}]) \right. \\ &\quad \left. + \partial_x V(\delta_i, S_i) Z_{n,i+1} - (\partial_x V(\delta_i, S_i))^+ \widehat{\mathbb{E}}[Z_{n,i+1} | \mathcal{H}_{n,i}] + (\partial_x V(\delta_i, S_i))^- \widehat{\mathcal{E}}[Z_{n,i+1} | \mathcal{H}_{n,i}] \right] \\ &\quad + \frac{1}{2} (\partial_{xx}^2 V(\delta_i, S_i))^- (ra_{n,i+1}^2 - \widehat{\mathcal{E}}[Z_{n,i+1}^2 | \mathcal{H}_{n,i}]) \\ &\quad + \left[(\partial_x V(\delta_i, S_i))^+ \widehat{\mathbb{E}}[Z_{n,i+1} | \mathcal{H}_{n,i}] - (\partial_x V(\delta_i, S_i))^- \widehat{\mathcal{E}}[Z_{n,i+1} | \mathcal{H}_{n,i}] \right] \\ &=: J_{n,1}^i + J_{n,2}^i + J_{n,3}^i \end{aligned}$$

and

$$\begin{aligned} I_n^i &= a_{n,i+1}^2 [(\partial_t V(\delta_i + \gamma a_{n,i+1}^2, S_{i+1}) - \partial_t V(\delta_i, S_{i+1})) + (\partial_t V(\delta_i, S_{i+1}) - \partial_t V(\delta_i, S_i))] \\ &\quad + \frac{1}{2} [\partial_{xx}^2 V(\delta_i, S_i + \beta Z_{n,i+1}) - \partial_{xx}^2 V(\delta_i, S_i)] Z_{n,i+1}^2 \\ &=: I_{n,1}^i + I_{n,2}^i, \end{aligned}$$

where γ and β are between 0 and 1. Thus

$$\widehat{\mathbb{E}} \left[\left| V(\delta_{k_n}, S_{k_n}) - V(0, 0) - \sum_{i=0}^{k_n-1} J_{n,1}^i \right| \right] \leq \widehat{\mathbb{E}} \left[\sum_{i=0}^{k_n-1} (|I_n^i| + |J_{n,2}^i| + |J_{n,3}^i|) \right].$$

For $J_{n,1}^i$, it follows that

$$\widehat{\mathbb{E}} [J_{n,1}^i | \mathcal{H}_{n,i}] = [\partial_t V(\delta_i, S_i) + G(\partial_{xx}^2 V(\delta_i, S_i))] a_{n,i+1}^2 = 0.$$

It follows that

$$\widehat{\mathbb{E}} \left[\sum_{i=0}^{k_n-1} J_{n,1}^i \right] = \widehat{\mathbb{E}} \left[\sum_{i=0}^{k_n-2} J_{n,1}^i + \widehat{\mathbb{E}} \left[J_{n,1}^{k_n-1} | \mathcal{H}_{n,k_n-1} \right] \right] = \widehat{\mathbb{E}} \left[\sum_{i=0}^{k_n-2} J_{n,1}^i \right] = \dots = 0. \quad (3.11)$$

For $J_{n,2}^i$, note $|\delta_i| \leq 1 + h/4$, and $|S_i| \leq 2N$ when $a_{n,i+1}^2 \neq 0$. We have

$$\left| \partial_{xx}^2 V(\delta_i, S_i) \right| \leq \sup_{0 \leq t \leq 1+h/4, |x| \leq 2N} \left| \partial_{xx}^2 V(t, x) \right| \leq C.$$

It follows that

$$\widehat{\mathbb{E}} \left[\sum_{i=0}^{k_n-1} |J_{n,2}^i| \right] \leq C \widehat{\mathbb{E}} \left[\sum_{i=1}^{k_n} |ra_{n,i}^2 - \widehat{\mathcal{E}}[Z_{n,i}^2 | \mathcal{H}_{n,i-1}]| \right] \rightarrow 0$$

by the facts that $\sum_{i=1}^{k_n} |ra_{n,i}^2 - \widehat{\mathcal{E}}[Z_{n,i}^2 | \mathcal{H}_{n,i-1}]| \leq \sum_{i=1}^{k_n} a_{n,i}^2 \leq 2$ and the condition (2.3). For $J_{n,3}^i$, note $|\partial_x V(t, x)| \leq C$. So

$$\widehat{\mathbb{E}} \left[\sum_{i=0}^{k_n-1} |J_{n,3}^i| \right] \leq C \widehat{\mathbb{E}}[\chi_{k_n}] \rightarrow 0$$

by the assumption that $\chi_{k_n} \leq 1$ and the condition (2.4).

For $I_{n,i}^i$, note both $\partial_t V$ and $\partial_{xx} V$ are uniformly α -Hölder continuous in x and $\alpha/2$ -Hölder continuous in t on $[0, 1 + h/2] \times R$. Without loss of generality, we assume $\alpha < \tau$. We then have

$$\begin{aligned} |I_{n,1}^i| &\leq C \left[|a_{n,i+1}|^{2+\alpha} + a_{n,i+1}^2 |Z_{n,i+1}|^\alpha \right] \\ &\leq C \left[|a_{n,i+1}|^{2+\alpha} + a_{n,i+1}^2 \epsilon^{\alpha/2} + a_{n,i+1}^2 [(Z_{n,i+1}^2 - \epsilon)^+]^{\alpha/2} \right] \\ &\leq C \left[|a_{n,i+1}|^{2+\alpha} + a_{n,i+1}^2 \epsilon^{\alpha/2} + |a_{n,i+1}|^{4/(2-\alpha)} + (Z_{n,i+1}^2 - \epsilon)^+ \right], \\ |I_{n,2}^i| &\leq C |Z_{n,i+1}|^{2+\alpha}, \text{ if } |Z_{n,i+1}| \leq 1, \end{aligned}$$

where the last inequality in $|I_{n,1}^i|$ is due to $|a|^{1-\alpha}|b|^{\alpha/2} \leq |a|(1 - \alpha/2) + |b|\alpha/2$. Note $|\partial_x V(t, x)| \leq C$ and $|\partial_{xx}^2 V(\delta_i, S_i)| \leq C$. On the other hand, when $|Z_{n,i+1}| \geq 1$,

$$\left| \frac{1}{2} \partial_{xx}^2 V(\delta_i, S_i + \beta Z_{n,i+1}) \right| = \left| \frac{V(\delta_i, S_{i+1}) - V(\delta_i, S_i)}{Z_{n,i+1}^2} - \frac{\partial_x V(\delta_i, S_i)}{Z_{n,i+1}} \right| \leq 2 + C.$$

It follows that $|I_{n,2}^i| \leq C Z_{n,i+1}^2$ if $|Z_{n,i+1}| \geq 1$. Hence

$$\begin{aligned} |I_n^i| &\leq |I_{n,1}^i| + C Z_{n,i+1}^2 (|Z_{n,i+1}|^\alpha \wedge 1) \leq |I_{n,1}^i| + C \epsilon^{\alpha/2} Z_{n,i+1}^2 + C (Z_{n,i+1}^2 - \epsilon)^+ \\ &= C \epsilon^{\alpha/2} a_{n,i+1}^2 + C |a_{n,i+1}|^{2+\alpha} + C |a_{n,i+1}|^{4/(2-\alpha)} + C \widehat{\mathbb{E}}[(Z_{n,i+1}^2 - \epsilon)^+ | \mathcal{H}_{n,i}] \\ &\quad + C \epsilon^{\alpha/2} (Z_{n,i+1}^2 - a_{n,i+1}^2) + C ((Z_{n,i+1}^2 - \epsilon)^+ - \widehat{\mathbb{E}}[(Z_{n,i+1}^2 - \epsilon)^+ | \mathcal{H}_{n,i}]), \end{aligned}$$

where the sub-linear expectations under $\widehat{\mathbb{E}}$ of the last two terms are zero. By noting $\sum_{k=1}^{k_n} \widehat{\mathbb{E}}[(Z_{n,i}^2 - \epsilon)^+ | \mathcal{H}_{n,i-1}] \leq \sum_{i=1}^{k_n} a_{n,i}^2 \leq 2$, the condition (2.1) and $\max_i |a_{n,i}| \xrightarrow{\mathbb{V}} 0$, it follows that

$$\begin{aligned} \widehat{\mathbb{E}} \left[\sum_{i=0}^{k_n-1} |I_n^i| \right] &\leq 2C\epsilon^\alpha + 2C\widehat{\mathbb{E}} \left[\max_i |a_{n,i}|^\alpha \right] + 2C\widehat{\mathbb{E}} \left[\max_i |a_{n,i}|^{2\alpha/(2-\alpha)} \right] \\ &\quad + C\widehat{\mathbb{E}} \left[\sum_{i=1}^{k_n} \widehat{\mathbb{E}}[(Z_{n,i}^2 - \epsilon)^+ | \mathcal{H}_{n,i-1}] \right] \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{and then } \epsilon \rightarrow 0. \end{aligned}$$

(3.10) is proved. Hence (3.6) holds for any bounded function φ with bounded derivative.

If φ is a bounded and uniformly continuous function, we define a function φ_δ as a convolution of φ and the density of a normal distribution $N(0, \delta)$, i.e.,

$$\varphi_\delta = \varphi * \psi_\delta, \quad \text{with } \psi_\delta(x) = \frac{1}{\sqrt{2\pi\delta}} \exp \left\{ -\frac{x^2}{2\delta} \right\}.$$

Then $|\varphi'_\delta(x)| \leq \sup_x |\varphi(x)|\delta^{-1/2}$ and $\sup_x |\varphi_\delta(x) - \varphi(x)| \rightarrow 0$ as $\delta \rightarrow 0$. Hence (3.6) holds for any bounded and uniformly continuous function φ .

Now, for a continuous function φ and a give a number $N > 1$, we define $\varphi_1(x) = \varphi((-N) \vee (x \wedge N))$. Then φ_1 is a bounded and uniformly continuous function, and $|\varphi(x) - \varphi_1(x)| \leq CI\{|x| > N\}$. And so

$$\begin{aligned} \sup_n \widehat{\mathbb{E}} \left[\left| \varphi \left(\sum_{k=1}^{k_n} Z_{n,k} \right) - \varphi_1 \left(\sum_{k=1}^{k_n} Z_{n,k} \right) \right| \right] &\leq C\mathbb{V} \left(\left| \sum_{k=1}^{k_n} Z_{n,k} \right| > N \right) \\ &\leq CN^{-2} \sup_n \widehat{\mathbb{E}} \left[\left(\sum_{k=1}^{k_n} Z_{n,k} \right)^2 \right] \leq CN^{-2} \sup_n \left(\widehat{\mathbb{E}}[\delta_{k_n}] + \widehat{\mathbb{E}}[\chi_{k_n}^2] \right) \\ &\leq 3CN^{-2} \rightarrow 0 \quad \text{as } N \rightarrow \infty \end{aligned}$$

by (3.5). The proof of Theorem 2.1 is now completed. \square

Proof of Theorem 2.2. Let $Z_{n,k} = X_{n,k}/B_n$. Then the conditions (2.1)-(2.4) are satisfied with $\rho = 1$. By Theorem 2.1, (2.11) holds for any bounded continuous function $\varphi(x)$. To show that it holds for a continuous function φ satisfying $\varphi(x) \leq C(1 + |x|^p)$. It is sufficient to show that the uniform continuity of $\{|\sum_{k=1}^{k_n} X_{n,k}|^p/B_n^p, n \geq 1\}$, i.e.,

$$\lim_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \widehat{\mathbb{E}} \left[\left(\left| \frac{\sum_{k=1}^{k_n} X_{n,k}}{B_n} \right|^p - N \right)^+ \right] = 0.$$

We show a more strong result that

$$\lim_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \widehat{\mathbb{E}} \left[\left(\max_{i \leq k_n} \left| \frac{\sum_{k=1}^i X_{n,k}}{B_n} \right|^p - N \right)^+ \right] = 0. \quad (3.12)$$

Let $Y_{n,k} = (-B_n) \vee X_{n,k} \wedge (B_n)$ and $\widehat{Y}_{n,k} = X_{n,k} - Y_{n,k}$. Then the Lindeberg condition (2.8) implies that

$$\begin{aligned} \frac{\sum_{k=1}^{k_n} \widehat{\mathbb{E}}[|\widehat{Y}_{n,k}|]}{B_n} &= \frac{\sum_{k=1}^{k_n} \widehat{\mathbb{E}}[(|X_{n,k}| - B_n)^+]}{B_n} \\ &\leq 2 \frac{\sum_{k=1}^{k_n} \widehat{\mathbb{E}}[(|X_{n,k}|^2 - B_n^2/2)^+]}{B_n^2} \rightarrow 0. \end{aligned} \quad (3.13)$$

It follows that

$$\frac{\sum_{k=1}^{k_n} \left\{ |\widehat{\mathbb{E}}[Y_{n,k}]| + |\widehat{\mathcal{E}}[Y_{n,k}]| \right\}}{B_n} \rightarrow 0. \quad (3.14)$$

Also, it is obvious that

$$\frac{\sum_{k=1}^{k_n} \widehat{\mathbb{E}}[|Y_{n,k}|^q]}{B_n^q} \leq \frac{B_n^{q-2} \sum_{k=1}^{k_n} \widehat{\mathbb{E}}[Y_{n,k}^2]}{B_n^q} \leq 1, \forall q \geq 2. \quad (3.15)$$

By Lemma 3.3,

$$\begin{aligned} \widehat{\mathbb{E}} \left[\max_{i \leq k_n} \left| \sum_{k=1}^i Y_{n,k} \right|^q \right] &\leq C_q \left\{ \sum_{k=1}^{k_n} \widehat{\mathbb{E}}[|Y_{n,k}|^q] + \left(\sum_{k=1}^{k_n} \widehat{\mathbb{E}}[Y_{n,k}^2] \right)^{q/2} \right. \\ &\quad \left. + \left(\sum_{k=1}^{k_n} \left(|\widehat{\mathbb{E}}[Y_{n,k}]| + |\widehat{\mathcal{E}}[Y_{n,k}]| \right) \right)^q \right\} \leq C_q B_n^{q/2}, \end{aligned}$$

by (3.14) and (3.15). It follows that

$$\begin{aligned} &\lim_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \widehat{\mathbb{E}} \left[\left(\max_{i \leq k_n} \left| \frac{\sum_{k=1}^i Y_{n,k}}{B_n} \right|^p - N \right)^+ \right] \\ &\leq \lim_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} N^{-2} \widehat{\mathbb{E}} \left[\max_{i \leq k_n} \left| \frac{\sum_{k=1}^i Y_{n,k}}{B_n} \right|^{2p} \right] = 0. \end{aligned}$$

For $\widehat{Y}_{n,k}$, by Lemma 3.3 we have

$$\begin{aligned}
& \widehat{\mathbb{E}} \left[\max_{i \leq k_n} \left| \sum_{k=1}^i \widehat{Y}_{n,k} \right|^p \right] \\
& \leq C_p \left\{ \frac{\sum_{k=1}^{k_n} \widehat{\mathbb{E}}[|\widehat{Y}_{n,k}|^p]}{B_n^p} + \left(\frac{\sum_{k=1}^{k_n} \widehat{\mathbb{E}}[|\widehat{Y}_{n,k}|^2]}{B_n^2} \right)^{p/2} + \left(\frac{\sum_{k=1}^{k_n} \widehat{\mathbb{E}}[|\widehat{Y}_{n,k}|]}{B_n} \right)^p \right\} \\
& \leq C_p \left\{ \frac{\sum_{k=1}^{k_n} \widehat{\mathbb{E}}[(|X_{n,k}|^p - B_n^p)^+]}{B_n^p} + \left(\frac{\sum_{k=1}^{k_n} \widehat{\mathbb{E}}[(X_{n,k}^2 - B_n^2)^+]}{B_n^2} \right)^{p/2} \right. \\
& \quad \left. + \left(\frac{\sum_{k=1}^{k_n} \widehat{\mathbb{E}}[(|X_{n,k}| - B_n)^+]}{B_n} \right)^p \right\} \\
& \rightarrow 0
\end{aligned}$$

by (3.13) and the condition (2.12). Thus (3.12) is verified and the proof is completed. \square .

At last, we prove the functional central limit theorems.

Proof of Theorem 2.3. For $0 \leq t \leq 1$ denote $i(t)$ be the integer i such that $\sum_{k=1}^i \bar{\sigma}_{n,k}^2 / B_n \leq t < \sum_{k=1}^{i+1} \bar{\sigma}_{n,k}^2 / B_n$. It is easily verified that for $t \geq 0$, $0 \leq t_1 < t_2 \leq 1$,

$$\frac{\sum_{k=1}^{i(t)} \bar{\sigma}_{n,k}^2}{B_n^2} \rightarrow t, \quad \frac{\sum_{k=i(t_1)+1}^{i(t_2)} \bar{\sigma}_{n,k}^2}{B_n^2} \rightarrow t_2 - t_1,$$

and the array of random variables $\{\frac{X_{n,i(t_1)+k}}{B_n}, \mathcal{H}_{n,i(t_1)+k}; k = 1, \dots, i(t_2) - i(t_1)\}$ satisfy the conditions (2.1)-(2.4) in Theorem 2.1 with $\rho = t_2 - t_1$. So

$$\frac{S_{n,i(t_2)} - S_{n,i(t_1)}}{B_n} \xrightarrow{d} W(t_2) - W(t_1).$$

Suppose $0 = t_1 < t_2 < \dots < t_l \leq 1$. Noting the independence, by Lemma 4.4 of Zhang (2015b) we have

$$\left(\frac{S_{n,i(t_2)} - S_{n,i(t_1)}}{B_n}, \dots, \frac{S_{n,i(t_l)} - S_{n,i(t_{l-1})}}{B_n} \right) \xrightarrow{d} (W(t_2) - W(t_1), \dots, W(t_l) - W(t_{l-1})).$$

So

$$\left(\frac{S_{n,i(t_1)}}{B_n}, \dots, \frac{S_{n,i(t_l)}}{B_n} \right) \xrightarrow{d} (W(t_1), \dots, W(t_l)),$$

which implies

$$(W_n(t_1), \dots, W_n(t_l)) \xrightarrow{d} (W(t_1), \dots, W(t_l)).$$

So, we have shown the convergence of finite dimensional distributions. By Theorem 9 of Peng (2010) on the tightness and the arguments of Zhang (2015b), to show that (2.16) holds

for bounded continuous function φ , it is sufficient to show that

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{V}(w_\delta(W_n) \geq 3\epsilon) = 0,$$

where $\omega_\delta(x) = \sup_{|t-s| < \delta, t, s \in [0,1]} |x(t) - x(s)|$. With the same argument of Billingsley (1968, Pages 56-59, c.f., (8.12)), it is sufficient to show that for $0 \leq t_1 < t_2 \leq 1$,

$$\limsup_{\lambda \rightarrow \infty} \limsup_{n \rightarrow \infty} \lambda^2 \mathbb{V} \left(\frac{\max_{k \leq i(t_2) - i(t_1)} |S_{n, i(t_1) + k} - S_{n, i(t_1)}|}{\sqrt{\sum_{k=i(t_1)+1}^{i(t_2)} \bar{\sigma}_{n,k}^2}} \geq \lambda \right) = 0.$$

Now,

$$\begin{aligned} & \lambda^2 \mathbb{V} \left(\frac{\max_{k \leq i(t_2) - i(t_1)} |S_{n, i(t_1) + k} - S_{n, i(t_1)}|}{\sqrt{\sum_{k=i(t_1)+1}^{i(t_2)} \bar{\sigma}_{n,k}^2}} \geq \lambda \right) \\ & \leq 2\hat{\mathbb{E}} \left[\left(\frac{\max_{k \leq i(t_2) - i(t_1)} |S_{n, i(t_1) + k} - S_{n, i(t_1)}|^2}{\sum_{k=i(t_1)+1}^{i(t_2)} \bar{\sigma}_{n,k}^2} - \frac{\lambda^2}{2} \right)^+ \right], \end{aligned}$$

which will converge to 0 as $n \rightarrow \infty$ firstly and then $\lambda \rightarrow \infty$, similarly to (3.12).

For a continuous function φ with $\varphi(x) \leq C(1 + \|x\|^p)$, it is sufficient to show that $\{\|W_n\|^p; n \geq 1\}$ are uniformly integrable under the condition (2.8) when $p = 2$ and the condition (2.12) when $p > 2$. Note

$$\|W_n\|^p = \max_{k \leq k_n} \left| \frac{\sum_{i=1}^k X_{n,i}}{B_n} \right|^p.$$

The uniform integrability follows from (3.12) and the proof is completed. \square

For showing Theorem 2.4, we need a more lemma.

Lemma 3.4 *Suppose that ξ_n and η_n are two independent random vectors in the sub-linear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, $\mathcal{H}_n = \{\varphi(\xi_n) : \varphi \in C_{l, \text{lip}}\}$. Suppose that \mathbf{X}_n is a d_1 -dimensional random vector and a local Lipschitz function of ξ_n and, \mathbf{Y}_n is a d_2 -dimensional random vector and a local Lipschitz function of (ξ_n, η_n) . Assume that $\mathbf{X}_n \xrightarrow{d} \mathbf{X}$ and for any bounded Lipschitz function $\varphi(\mathbf{x}, \mathbf{y}) : \mathbb{R}_{d_1} \otimes \mathbb{R}_{d_2} \rightarrow \mathbb{R}$,*

$$\hat{\mathbb{E}} \left[\left| \hat{\mathbb{E}}[\varphi(\mathbf{x}, \mathbf{Y}_n) | \mathcal{H}_n] - \tilde{\mathbb{E}}[\varphi(\mathbf{x}, \mathbf{Y})] \right| \right] \rightarrow 0 \quad \forall \mathbf{x}, \quad (3.16)$$

where \mathbf{X}, \mathbf{Y} are two random vectors in a sub-linear expectation space $(\Omega, \mathcal{H}, \tilde{\mathbb{E}})$ with $\tilde{\mathbb{V}}(\|\mathbf{X}\| > \lambda) \rightarrow 0$ and $\tilde{\mathbb{V}}(\|\mathbf{Y}\| > \lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$. Then

$$(\mathbf{X}_n, \mathbf{Y}_n) \xrightarrow{d} (\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}), \quad (3.17)$$

where $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$ are independent, $\tilde{\mathbf{X}} \stackrel{d}{=} \mathbf{X}$ and $\tilde{\mathbf{Y}} \stackrel{d}{=} \mathbf{Y}$.

Proof. Suppose $\varphi(\mathbf{x}, \mathbf{y}) : \mathbb{R}_{d_1} \otimes \mathbb{R}_{d_2} \rightarrow \mathbb{R}$ is a bounded continuous function. We want to show that

$$\widehat{\mathbb{E}}[\varphi(\mathbf{X}_n, \mathbf{Y}_n)] \rightarrow \widetilde{\mathbb{E}}[\varphi(\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}})]. \quad (3.18)$$

First we assume that $\varphi(\mathbf{x}, \mathbf{y})$ is a bounded Lipschitz function. Let $g_n(\mathbf{x}) = \widehat{\mathbb{E}}[\varphi(\mathbf{x}, \mathbf{Y}_n) | \mathcal{H}_n]$ and $g(\mathbf{x}) = \widetilde{\mathbb{E}}[\varphi(\mathbf{x}, \mathbf{Y})]$. It is easily seen that

$$\widehat{\mathbb{E}}[\varphi(\mathbf{X}_n, \mathbf{Y}_n)] = \widehat{\mathbb{E}}[g_n(\mathbf{X}_n)], \quad (3.19)$$

$$\widetilde{\mathbb{E}}[\varphi(\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}})] = \widetilde{\mathbb{E}}[g(\mathbf{X})].$$

For any sequence $\{\mathbf{x}_n\}$ with $\mathbf{x}_n \rightarrow \mathbf{x}$, we have

$$\begin{aligned} \widehat{\mathbb{E}}[|g_n(\mathbf{x}_n) - g(\mathbf{x})|] &= \widehat{\mathbb{E}}\left[\left|\widehat{\mathbb{E}}[\varphi(\mathbf{x}_n, \mathbf{Y}_n) | \mathcal{H}_n] - \widetilde{\mathbb{E}}[\varphi(\mathbf{x}, \mathbf{Y})]\right|\right] \\ &\leq \widehat{\mathbb{E}}\left[\left|\widehat{\mathbb{E}}[\varphi(\mathbf{x}_n, \mathbf{Y}_n) | \mathcal{H}_n] - \widehat{\mathbb{E}}[\varphi(\mathbf{x}, \mathbf{Y}_n) | \mathcal{H}_n]\right| + \left|\widehat{\mathbb{E}}[\varphi(\mathbf{x}, \mathbf{Y}_n) | \mathcal{H}_n] - \widetilde{\mathbb{E}}[\varphi(\mathbf{x}, \mathbf{Y})]\right|\right] \\ &\leq \sup_{\mathbf{y}} |\varphi(\mathbf{x}_n, \mathbf{y}) - \varphi(\mathbf{x}, \mathbf{y})| + \widehat{\mathbb{E}}\left[\left|\widehat{\mathbb{E}}[\varphi(\mathbf{x}, \mathbf{Y}_n) | \mathcal{H}_n] - \widetilde{\mathbb{E}}[\varphi(\mathbf{x}, \mathbf{Y})]\right|\right] \\ &\rightarrow 0 \end{aligned} \quad (3.20)$$

by noting that $\varphi(\mathbf{x}, \mathbf{y})$ is uniformly continuous and (3.16). It follows that

$$\sup_{\|\mathbf{x}\| \leq \lambda} \widehat{\mathbb{E}}[|g_n(\mathbf{x}) - g(\mathbf{x})|] \rightarrow 0.$$

Hence

$$\begin{aligned} \left|\widehat{\mathbb{E}}[\varphi(\mathbf{X}_n, \mathbf{Y}_n)] - \widetilde{\mathbb{E}}[\varphi(\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}})]\right| &= \left|\widehat{\mathbb{E}}[g_n(\mathbf{X}_n)] - \widetilde{\mathbb{E}}[g(\mathbf{X})]\right| \\ &\leq \left|\widehat{\mathbb{E}}[g_n(\mathbf{X}_n)] - \widehat{\mathbb{E}}[g(\mathbf{X}_n)]\right| + \left|\widehat{\mathbb{E}}[g(\mathbf{X}_n)] - \widetilde{\mathbb{E}}[g(\mathbf{X})]\right| \\ &\leq \sup_{\|\mathbf{x}\| \leq \lambda} \widehat{\mathbb{E}}[|g_n(\mathbf{x}) - g(\mathbf{x})|] + 2M\mathbb{V}(|\mathbf{X}_n| \geq \lambda) + \left|\widehat{\mathbb{E}}[g(\mathbf{X}_n)] - \widetilde{\mathbb{E}}[g(\mathbf{X})]\right|. \end{aligned}$$

Choose a Lipschitz function $f(x)$ such that $I\{x > \lambda\} \geq f(x) \geq I\{x > \lambda/2\}$. Letting $n \rightarrow \infty$ yields that

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \left|\widehat{\mathbb{E}}[\varphi(\mathbf{X}_n, \mathbf{Y}_n)] - \widetilde{\mathbb{E}}[\varphi(\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}})]\right| \\ &\leq 2M \limsup_{n \rightarrow \infty} \mathbb{V}(\|\mathbf{X}_n\| > \lambda) \leq 2M \limsup_{n \rightarrow \infty} \widehat{\mathbb{E}}[f(\|\mathbf{X}_n\|)] \\ &= 2M \widetilde{\mathbb{E}}[f(\|\mathbf{X}\|)] \leq 2M \widetilde{\mathbb{V}}(\|\mathbf{X}\| > \lambda/2) \rightarrow 0 \text{ as } \lambda \rightarrow \infty. \end{aligned}$$

So, (3.18) is proved for a bounded and uniformly continuous function. Finally, let $\varphi(\mathbf{x}, \mathbf{y})$ be a bounded continuous function with $|\varphi(\mathbf{x}, \mathbf{y})| \leq M$. Let $\lambda > 0$. For $\mathbf{x} = (x_1, \dots, x_d)$,

denote $\mathbf{x}_\lambda = ((-\lambda) \vee (x_1 \wedge \lambda)\lambda, \dots, (-\lambda) \vee (x_{d_1} \wedge \lambda))$ and define \mathbf{y}_λ similarly. Let $\varphi_\lambda(\mathbf{x}, \mathbf{y}) = \varphi(\mathbf{x}_\lambda, \mathbf{y}_\lambda)$. Then φ_λ is a bounded uniformly continuous function with

$$|\varphi_\lambda(\mathbf{x}, \mathbf{y}) - \varphi(\mathbf{x}, \mathbf{y})| \leq 2MI\{\|\mathbf{x}\| > \lambda\} + 2MI\{\|\mathbf{y}\| > \lambda\}.$$

It follows that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left| \widehat{\mathbb{E}}[\varphi(\mathbf{X}_n, \mathbf{Y}_n)] - \widetilde{\mathbb{E}}[\varphi(\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}})] \right| \\ & \leq \limsup_{n \rightarrow \infty} \left| \widehat{\mathbb{E}}[\varphi_\lambda(\mathbf{X}_n, \mathbf{Y}_n)] - \widetilde{\mathbb{E}}[\varphi_\lambda(\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}})] \right| \\ & \quad + 2M \limsup_{n \rightarrow \infty} \{ \mathbb{V}(\|\mathbf{X}_n\| > \lambda) + \mathbb{V}(\|\mathbf{Y}_n\| > \lambda) \} \\ & \quad + 2M \{ \widetilde{\mathbb{V}}(\|\mathbf{X}\| > \lambda) + \widetilde{\mathbb{V}}(\|\mathbf{Y}\| > \lambda) \} \\ & \leq 4M \{ \widetilde{\mathbb{V}}(\|\mathbf{X}\| > \lambda/2) + \widetilde{\mathbb{V}}(\|\mathbf{Y}\| > \lambda/2) \} \rightarrow 0 \text{ as } \lambda \rightarrow \infty. \end{aligned}$$

The proof is completed. \square

Proof of Theorem 2.4. Let $0 < t_1 < t_2 \leq 1$. Consider $\{Z_{n,k}^* =: Z_{n,[k_n t_1]+k}; k = 1, \dots, k_n^*\}$, $S_i^* = \sum_{k=1}^i Z_{n,[k_n t_1]+k}$, $k_n^* = [k_n t_2] - [k_n t_1]$. Then $S_{k_n^*}^* = S_{n,[k_n t_2]} - S_{n,[k_n t_1]} = \sum_{k=1}^{[k_n t_2]-[k_n t_1]} Z_{n,[k_n t_1]+k}$,

$$\sum_{k=1}^{k_n^*} \widehat{\mathbb{E}} \left[Z_{n,[k_n t_1]+k}^2 \middle| \mathcal{H}_{n,[k_n t_1]+k-1} \right] \xrightarrow{\mathbb{V}} \rho(t_2) - \rho(t_1).$$

By Theorem 2.2,

$$S_{n,[k_n t_2]} - S_{n,[k_n t_1]} \xrightarrow{d} W(\rho(t_2)) - W(\rho(t_1)).$$

Further, for any a bounded Lipschitz function $\varphi(\mathbf{u}, x)$, let $V^{\mathbf{u}}(t, x)$ be the unique viscosity solution of the following equation,

$$\partial_t V^{\mathbf{u}} + G(\partial_{xx}^2 V^{\mathbf{u}}) = 0, \quad (t, x) \in [0, 1+h] \times \mathbb{R}, \quad V^{\mathbf{u}}|_{t=1+h} = \varphi(\mathbf{u}, x).$$

With the same argument for showing (3.6), we can show that

$$\widehat{\mathbb{E}} \left[\left| \widehat{\mathbb{E}} [\varphi(\mathbf{u}, S_{n,[k_n t_2]} - S_{n,[k_n t_1]}) \middle| \mathcal{H}_{n,[k_n t_1]}] - \widetilde{\mathbb{E}} [\varphi(\mathbf{u}, W(\rho(t_2)) - W(\rho(t_1)))] \right| \right] \rightarrow 0. \quad (3.21)$$

The only difference is that (3.11) is needed to be replaced by

$$\begin{aligned}
& \widehat{\mathbb{E}} \left[\sum_{i=0}^{k_n^*-1} J_{n,1,*}^i \middle| \mathcal{H}_{n,[k_n t_1]} \right] \\
&= \widehat{\mathbb{E}} \left[\widehat{\mathbb{E}} \left[\sum_{i=0}^{k_n^*-1} J_{n,1,*}^i \middle| \mathcal{H}_{n,[k_n t_1]+k_n^*-1} \right] \middle| \mathcal{H}_{n,[k_n t_1]} \right] \\
&= \widehat{\mathbb{E}} \left[\sum_{i=0}^{k_n^*-2} J_{n,1,*}^i + \widehat{\mathbb{E}} \left[J_{n,1,*}^{k_n^*-1} \middle| \mathcal{H}_{n,[k_n t_1]+k_n^*-1} \right] \middle| \mathcal{H}_{n,[k_n t_1]} \right] \\
&= \widehat{\mathbb{E}} \left[\sum_{i=0}^{k_n^*-2} J_{n,1,*}^i \middle| \mathcal{H}_{n,[k_n t_1]} \right] = \dots = 0,
\end{aligned} \tag{3.22}$$

where $J_{n,1,*}^i$ is defined the same as $J_{n,1}^i$ with $\{Z_{n,k}^*\}$ taking the place of $\{Z_{n,k}\}$. On the other hand, note

$$S_{n,[nt_1]} \xrightarrow{d} W(\rho(t_1)).$$

Hence,

$$\left(S_{n,[nt_1]}, S_{n,[nt_2]} - S_{n,[nt_1]} \right) \xrightarrow{d} \left(W(\rho(t_1)), W(\rho(t_2)) - W(\rho(t_1)) \right).$$

by (3.21) and Lemma 3.4. By induction, for any $0 = t_0 < \dots < t_d < 1$,

$$\begin{aligned}
& \left(S_{n,[nt_1]} - S_{n,[nt_0]}, \dots, S_{n,[nt_d]} - S_{n,[nt_{d-1}]} \right) \\
& \xrightarrow{d} \left(W(\rho(t_1)) - W(\rho(t_0)), \dots, W(\rho(t_d)) - W(\rho(t_{d-1})) \right),
\end{aligned}$$

which implies

$$\left(W_n(t_1), \dots, W_n(t_d) \right) \xrightarrow{d} \left(W(\rho(t_1)), \dots, W(\rho(t_d)) \right).$$

Next, it is sufficient to show that for any $\epsilon' > 0$,

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{V}(w_\delta(W_n) \geq 3\epsilon') = 0. \tag{3.23}$$

Assume $0 < \delta < 1/10$. Let $0 = t_0 < t_1 \dots < t_K = 1$ such that $t_k - t_{k-1} = \delta$, and let $t_{K+1} = t_{K+2} = 1$. It is easily seen that

$$\mathbb{V}(w_\delta(W_n) \geq 3\epsilon') \leq 2 \sum_{k=0}^{K-1} \mathbb{V} \left(\max_{s \in [t_k, t_{k+2}]} |S_{n,[k_n s]} - S_{n,[k_n t_k]}| \geq \epsilon' \right),$$

where $\omega_\delta(x) = \sup_{|t-s| < \delta, t, s \in [0,1]} |x(t) - x(s)|$. With the same argument as that at the beginning of the proof of Theorem 2.1, we can assume that $\delta_{k_n} = \sum_{k=1}^{k_n} \widehat{\mathbb{E}}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}] \leq 2\rho(1)$ and $\chi_{k_n} =: \sum_{k=1}^{k_n} \left\{ |\widehat{\mathbb{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}]| + |\widehat{\mathcal{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}]| \right\} < 1$. For any $\epsilon > 0$, let $Z_{n,k,1} =$

$(-\epsilon) \wedge Z_{n,k} \vee \epsilon$, $Z_{n,k,2} = Z_{n,k} - Z_{n,k,1}$, $S_{n,i,j} = \sum_{k=1}^i Z_{n,k,j}$, $j = 1, 2$. Then for $t, \tau > 0$, by (3.2),

$$\begin{aligned}
& \widehat{\mathbb{E}} \left[\max_{s \leq \tau} |S_{n,[k_n(t+s)],2} - S_{n,[k_n t],2}|^2 \right] \\
& \leq C \widehat{\mathbb{E}} \left[\sum_{k=[k_n t]+1}^{[k_n(t+\tau)]} \widehat{\mathbb{E}} [Z_{n,k,2}^2 | \mathcal{H}_{n,k-1}] \right] \\
& \quad + C \widehat{\mathbb{E}} \left[\left(\sum_{k=[k_n t]+1}^{[k_n(t+\tau)]} \left\{ |\widehat{\mathbb{E}}[Z_{n,k,2} | \mathcal{H}_{n,k-1}]| + |\widehat{\mathcal{E}}[Z_{n,k,2} | \mathcal{H}_{n,k-1}]| \right\} \right)^2 \right] \\
& \leq C \widehat{\mathbb{E}} \left[\sum_{k=[k_n t]+1}^{[k_n(t+\tau)]} \widehat{\mathbb{E}} [(Z_{n,k}^2 - \epsilon^2)^+ | \mathcal{H}_{n,k-1}] \right] \\
& \quad + C \epsilon^{-2} \widehat{\mathbb{E}} \left[\left(\sum_{k=[k_n t]+1}^{[k_n(t+\tau)]} \widehat{\mathbb{E}} [(Z_{n,k}^2 - \epsilon^2/2)^+ | \mathcal{H}_{n,k-1}] \right)^2 \right] \\
& \rightarrow 0 \text{ uniformly in } t, \tau,
\end{aligned}$$

by (2.1). For $Z_{n,k,1}$, by (3.3) we have

$$\begin{aligned}
& \widehat{\mathbb{E}} \left[\max_{s \leq \tau} |S_{n,[k_n(t+s)],1} - S_{n,[k_n t],1}|^4 \right] \\
& \leq C \widehat{\mathbb{E}} \left[\sum_{k=[k_n t]+1}^{[k_n(t+\tau)]} \widehat{\mathbb{E}} [Z_{n,k,1}^4 | \mathcal{H}_{n,k-1}] \right] + C \widehat{\mathbb{E}} \left[\left(\sum_{k=[k_n t]+1}^{[k_n(t+\tau)]} \widehat{\mathbb{E}} [Z_{n,k,1}^2 | \mathcal{H}_{n,k-1}] \right)^2 \right] \\
& \quad + C \widehat{\mathbb{E}} \left[\left(\sum_{k=[k_n t]+1}^{[k_n(t+\tau)]} \left\{ |\widehat{\mathbb{E}}[Z_{n,k,1} | \mathcal{H}_{n,k-1}]| + |\widehat{\mathcal{E}}[Z_{n,k,1} | \mathcal{H}_{n,k-1}]| \right\} \right)^4 \right] \\
& \leq C \epsilon^2 \widehat{\mathbb{E}} \left[\sum_{k=[k_n t]+1}^{[k_n(t+\tau)]} \widehat{\mathbb{E}} [Z_{n,k}^2 | \mathcal{H}_{n,k-1}] \right] + C \widehat{\mathbb{E}} \left[\left(\sum_{k=[k_n t]+1}^{[k_n(t+\tau)]} \widehat{\mathbb{E}} [Z_{n,k}^2 | \mathcal{H}_{n,k-1}] \right)^2 \right] \\
& \quad + C \widehat{\mathbb{E}} \left[\left(\sum_{k=1}^{k_n} \left\{ |\widehat{\mathbb{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}]| + |\widehat{\mathcal{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}]| \right\} \right)^4 \right] \\
& \quad + C \widehat{\mathbb{E}} \left[\epsilon^{-4} \left(\sum_{k=1}^{k_n} \widehat{\mathbb{E}} [(Z_{n,k}^2 - \epsilon^2/2)^+ | \mathcal{H}_{n,k-1}] \right)^4 \right].
\end{aligned}$$

The last two terms above will go to zero by (2.1) and (2.4). For considering the first two terms, we note

$$2\rho(1) \geq \sum_{k=[k_n t]+1}^{[k_n(t+\tau)]} \widehat{\mathbb{E}} [Z_{n,k}^2 | \mathcal{H}_{n,k-1}] \xrightarrow{\mathbb{V}} \rho(t+\tau) - \rho(t).$$

It follows that

$$\widehat{\mathbb{E}} \left[\sum_{k=[k_n t]+1}^{[k_n(t+\tau)]} \widehat{\mathbb{E}} [Z_{n,k}^2 | \mathcal{H}_{n,k-1}] \right] \rightarrow \rho(t+\tau) - \rho(t)$$

and

$$\widehat{\mathbb{E}} \left[\left(\sum_{k=[k_n t]+1}^{[k_n(t+\tau)]} \widehat{\mathbb{E}} [Z_{n,k}^2 | \mathcal{H}_{n,k-1}] \right)^2 \right] \rightarrow (\rho(t+\tau) - \rho(t))^2.$$

So, we conclude that

$$\begin{aligned} & \limsup_n 2 \sum_{k=0}^{K-1} \mathbb{V} \left(\max_{s \in [t_k, t_{k+2}]} |S_{n,[k_n s]} - S_{n,[k_n t_k]}| \geq \epsilon' \right) \\ & \leq \limsup_n 2 \sum_{k=0}^{K-1} \left(\frac{2}{\epsilon^*} \right)^2 \widehat{\mathbb{E}} \left[\max_{s \in [t_k, t_{k+2}]} |S_{n,[k_n s]} - S_{n,[k_n t_k]}|^2 \right] \\ & \quad + \limsup_n 2 \sum_{k=0}^{K-1} \left(\frac{2}{\epsilon^*} \right)^4 \sup_t \widehat{\mathbb{E}} \left[\max_{s \in [t_k, t_{k+2}]} |S_{n,[k_n s]} - S_{n,[k_n t_k]}|^4 \right] \\ & \leq C \sum_{k=0}^{K-1} \frac{1}{(\epsilon^*)^4} \left(\epsilon^2 (\rho(t_{k+2}) - \rho(t_k)) + (\rho(t_{k+2}) - \rho(t_k))^2 \right) \\ & \leq C \frac{\rho(1)}{(\epsilon^*)^4} \left(\epsilon^2 + \sup_{|t-s| \leq 2\delta} |\rho(t) - \rho(s)| \right) \rightarrow 0 \end{aligned}$$

by taking $\delta \rightarrow 0$ and $\epsilon \rightarrow 0$. Hence (3.23) is verified. And the proof is completed. \square

4 Generalization and Lévy characterization of a G-Brownian motion.

In this section, we consider a general martingale. Let $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ be a sub-linear expectation space, $\mathcal{H}_{n,0} \subset \dots \subset \mathcal{H}_{n,k_n}$ be subspaces of \mathcal{H} such that (i) any constant $c \in \mathcal{H}_{n,k}$ and, (ii) if $X_1, \dots, X_d \in \mathcal{H}_{n,k}$, then $\varphi(X_1, \dots, X_d) \in \mathcal{H}_{n,k}$ for any $\varphi \in C_{l, \text{lip}}(\mathbb{R}_d)$, $k = 0, \dots, k_n$. Denote $\mathcal{L}(\mathcal{H}) = \{X : \widehat{\mathbb{E}}[|X|] < \infty, X \in \mathcal{H}\}$. We consider a system of operators in $\mathcal{L}(\mathcal{H})$,

$$\widehat{\mathbb{E}}_{n,k} : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{H}_{n,k}$$

and denote $\widehat{\mathbb{E}}[X | \mathcal{H}_{n,k}] = \widehat{\mathbb{E}}_{n,k}[X]$, $\widehat{\mathcal{E}}[X | \mathcal{H}_{n,k}] = -\widehat{\mathbb{E}}_{n,k}[-X]$. Suppose that the operators $\widehat{\mathbb{E}}_{n,k}$ satisfy the following properties: for all $X, Y \in \mathcal{L}(\mathcal{H})$,

- (a) If $X \geq Y$ then $\widehat{\mathbb{E}}_{n,k}[X] \geq \widehat{\mathbb{E}}_{n,k}[Y]$;
- (b) $\widehat{\mathbb{E}}_{n,k}[X + Y] = X + \widehat{\mathbb{E}}_{n,k}[Y]$ if $X \in \mathcal{H}_{n,k}$, and $\widehat{\mathbb{E}}_{n,k}[XY] = X^+ \widehat{\mathbb{E}}_{n,k}[Y] - X^- \widehat{\mathbb{E}}_{n,k}[-Y]$
 $X \in \mathcal{H}_{n,k}$ and $XY \in \mathcal{L}(\mathcal{H})$;
- (c) $\widehat{\mathbb{E}}[\widehat{\mathbb{E}}_{n,k}[X]] = \widehat{\mathbb{E}}[X]$;
- (d) $\widehat{\mathbb{E}}_{n,k}[X] - \widehat{\mathbb{E}}_{n,k}[Y] \leq \widehat{\mathbb{E}}_{n,k}[X - Y]$;

$$(e) \quad \widehat{\mathbb{E}}_{n,l} \left[\left[\widehat{\mathbb{E}}_{n,k}[X] \right] \right] = \widehat{\mathbb{E}}_{n,l \wedge k}[X];$$

(f) If $\mathbf{X} = (X_1, \dots, X_d) \in \mathcal{H}_{n,k}$, $Z \in \mathcal{H}$ and $\varphi(\mathbf{x}, y)$ is a bounded Lipschitz function, then

$$\widehat{\mathbb{E}}[\varphi(\mathbf{X}, Z)] = \widehat{\mathbb{E}} \left[\widehat{\mathbb{E}}_{n,k}[\varphi(\mathbf{x}, Z)] \Big|_{\mathbf{x}=\mathbf{X}} \right].$$

It is easily seen that (b) implies that $\widehat{\mathbb{E}}_{n,k}[c] = c$ and $\widehat{\mathbb{E}}_{n,k}[\lambda X] = \lambda \widehat{\mathbb{E}}_{n,k}[X]$ if $\lambda \geq 0$.

Theorem 4.1 *Suppose that the operators $\widehat{\mathbb{E}}_{n,k}$ satisfy (a)-(c), $\{Z_{n,k}; k = 1, \dots, k_n\}$ is an array of random variables such that $Z_{n,k} \in \mathcal{H}_{n,k}$ and $\widehat{\mathbb{E}}[Z_{n,k}^2] < \infty$, $k = 1, \dots, k_n$. Assume that the conditions (2.1)-(2.4) in Theorem 2.1 are satisfied. Then*

$$\sum_{k=1}^{k_n} Z_{n,k} \xrightarrow{d} N(0, [r\rho, \rho]).$$

Further, assume that the operators $\widehat{\mathbb{E}}_{n,k}$ also satisfy (d)-(f), and the condition (2.17) is also satisfied. Then the conclusion in Theorem 2.4 holds.

Proof. The proof is similar to that of Theorems 2.1 and Theorem 2.4, where the condition (f) is used to verify (3.19) with $\mathbf{X}_n = (S_{n,[k_n t_1]} - S_{n,[k_n t_0]}, \dots, S_{n,[k_n t_{d-1}]} - S_{n,[k_n t_{d-2}]})$ and $\mathbf{Y}_n = S_{n,[k_n t_d]} - S_{n,[k_n t_{d-1}]}$, the condition (d) is used to verify (3.20) and the condition (e) is used to verify (3.22). \square

Now, as an application of Theorem 4.1 we will give a Lévy characterization of a G-Brownian motion. Let $\{\mathcal{H}_t; t \geq 0\}$ be a non-decreasing family of subspaces of \mathcal{H} such that (i) a constant $c \in \mathcal{H}_{n,k}$ and, (ii) if $X_1, \dots, X_d \in \mathcal{H}_t$, then $\varphi(X_1, \dots, X_d) \in \mathcal{H}_t$ for any $\varphi \in C_{l, lip}$. We consider a system of operators in $\mathcal{L}(\mathcal{H})$,

$$\widehat{\mathbb{E}}_t : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{H}_t$$

and denote $\widehat{\mathbb{E}}[X|\mathcal{H}_t] = \widehat{\mathbb{E}}_t[X]$, $\widehat{\mathcal{E}}[X|\mathcal{H}_t] = -\widehat{\mathbb{E}}_t[-X]$. Suppose that the operators $\widehat{\mathbb{E}}_t$ satisfy the following properties: for all $X, Y \in \mathcal{L}(\mathcal{H})$,

$$(i) \quad \text{If } X \geq Y \text{ then } \widehat{\mathbb{E}}_t[X] \geq \widehat{\mathbb{E}}_t[Y];$$

$$(ii) \quad \widehat{\mathbb{E}}_t[X] - \widehat{\mathbb{E}}_t[Y] \leq \widehat{\mathbb{E}}_t[X - Y];$$

$$(iii) \quad \widehat{\mathbb{E}}_t[X + Y] = X + \widehat{\mathbb{E}}_t[Y] \text{ if } X \in \mathcal{H}_t, \text{ and } \widehat{\mathbb{E}}_t[XY] = X^+ \widehat{\mathbb{E}}_t[Y] - X^- \widehat{\mathbb{E}}_t[-Y] \text{ if } X \in \mathcal{H}_t \\ \text{and } XY \in \mathcal{L}(\mathcal{H});$$

$$(iv) \quad \widehat{\mathbb{E}} \left[\widehat{\mathbb{E}}_t[X] \right] = \widehat{\mathbb{E}}[X];$$

$$(v) \quad \widehat{\mathbb{E}}_t \left[\left[\widehat{\mathbb{E}}_s[X] \right] \right] = \widehat{\mathbb{E}}_{t \wedge s}[X];$$

(vi) If $\mathbf{X} = (X_1, \dots, X_d) \in \mathcal{H}_t$, $Z \in \mathcal{H}$ and $\varphi(\mathbf{x}, y)$ is a bounded Lipschitz function, then

$$\widehat{\mathbb{E}}[\varphi(\mathbf{X}, Z)] = \widehat{\mathbb{E}} \left[\widehat{\mathbb{E}}_t [\varphi(\mathbf{x}, Z)] \Big|_{\mathbf{x}=\mathbf{X}} \right].$$

Example 4.1 Let W_t be a G -Brownian motion in a sub-linear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$, and

$$\widetilde{\mathcal{H}} = \{X = \varphi(W_{t_1}, \dots, W_{t_d}) : 0 \leq t_1 \leq \dots \leq t_d, \varphi \in C_{l,Lip}(\mathbb{R}_d), d \geq 1\},$$

$$\mathcal{H}_t = \{X = \varphi(W_{t_1}, \dots, W_{t_d}) : 0 \leq t_1 \leq \dots \leq t_d \leq t, \varphi \in C_{l,Lip}(\mathbb{R}_d), d \geq 1\}.$$

For a $X = \varphi(W_{t_1}, \dots, W_{t_d}) \in \widetilde{\mathcal{H}}$, assume $0 \leq t_1 \leq t_i \leq t \leq t_{i+1} \leq \dots \leq t_d$, and define

$$\widehat{\mathbb{E}}_t[X] = \widehat{\mathbb{E}} \left[\varphi(w_{t_1}, \dots, w_{t_i}, W_{t_{i+1}} - W_t + w_t, \dots, W_{t_d} - W_t + w_t) \Big|_{w_{t_1}=W_{t_1}, \dots, w_{t_i}=W_{t_i}, w_t=W_t} \right].$$

Then in the sub-linear expectation space $(\Omega, \widetilde{\mathcal{H}}, \widehat{\mathbb{E}})$, the family $\{\mathcal{H}_t, \widehat{\mathbb{E}}_t\}_{t \geq 0}$ satisfies the properties (i)-(vi).

Definition 4.1 A process M_t is called a martingale if $M_t \in \mathcal{L}(\mathcal{H})$, $M_t \in \mathcal{H}_t$ and

$$\widehat{\mathbb{E}}[M_t | \mathcal{H}_s] = M_s, \quad s \leq t.$$

The following theorem gives a Lévy characterization of a G -Brownian motion.

Theorem 4.2 Let M_t be a random process in $(\Omega, \mathcal{H}, \mathcal{H}_t, \widehat{\mathbb{E}})$ with $M_0 = 0$,

$$\widehat{\mathbb{E}}[|M_t|^p] \leq C_V(|M_t|^p) \quad \text{for all } p > 0 \text{ and } t \geq 0. \quad (4.1)$$

Suppose that M_t satisfies

(I) Both M_t and $-M_t$ are martingales;

(II) For a constant $\overline{\sigma}^2 > 0$, $M_t^2 - \overline{\sigma}^2 t$ is a martingale;

(III) For a constant $0 < \underline{\sigma}^2 \leq \overline{\sigma}^2$, $-(M_t^2 - \underline{\sigma}^2 t)$ is a martingale;

(IV) There is a constant $\delta > 0$ such that for any $t > s > 0$, $\widehat{\mathbb{E}}[|M_t - M_s|^{2+\delta}] = o(t - s)$ as

$t - s \rightarrow 0$, or

(IV') For any $T, \epsilon > 0$,

$$\lim_{\delta \rightarrow 0} \mathbb{V}(w_T(M, \delta) > \epsilon) = 0, \text{ where } w_T(M, \delta) = \sup_{|t-s| < \delta, t, s \in [0, T]} |M(t) - M(s)|.$$

Then M_t is a G -Brownian motion with $M_1 \sim N(0, [\underline{\sigma}^2, \bar{\sigma}^2])$.

Remark 4.1 The Lévy characterization of a G -Brownian motion is first established by Xu and Zhang (2009, 2010) by using the stochastic calculus. An elementary proof of the Lévy characterization of a classical Brownian motion without using stochastic calculus appears in Doob (1953, c.f. Theorem 11.9). The stopping-time method is used for estimating the moments of a martingale in Doob (1953). In the framework of the sub-linear expectation, we shall avoid to use the stopping time because a stopped martingale may be not a martingale due the non-additive of the sub-linear expectation. We will give an elementary proof by using the functional central limit theorem.

Remark 4.2 The assumption (I) implies that $\widehat{\mathbb{E}}[M_t - M_s | \mathcal{H}_s] = \widehat{\mathcal{E}}[M_t - M_s | \mathcal{H}_s] = 0$ for all $t > s$. Also under the assumptions (I), the assumption (II) is equivalent to that $\widehat{\mathbb{E}}[(M_t - M_s)^2 | \mathcal{H}_s] = \bar{\sigma}^2(t - s)$ for all $t > s$, (III) is equivalent to that $\widehat{\mathcal{E}}[(M_t - M_s)^2 | \mathcal{H}_s] = \underline{\sigma}^2(t - s)$ for all $t > s$.

The each assumption of (IV) and (IV') means that M_t is continuous, but in different senses. The assumption (IV) means that M_t is continuous in $L_{2+\delta}$ at each time t , while, (IV') means that M_t is continuous in capacity \mathbb{V} uniformly in t on each finite interval.

Remark 4.3 In the framework of Xu and Zhang (2009, 2010), (vi) is not assumed for the operators $\widehat{\mathbb{E}}_t$. However, in the Step 5 of the proof in Xu and Zhang (2009) and the Step 4 of the proof in Xu and Zhang (2010), such a property is used. In fact, the following equality stated in Xu and Zhang (2009, page 242; 2010, page 2065) is just the property (vi),

$$\widehat{\mathbb{E}}[\varphi(M_{t_1}, M_{t_2} - M_{t_1})] = \widehat{\mathbb{E}} \left[\widehat{\mathbb{E}}[\varphi(x, M_{t_2} - M_{t_1}) | \mathcal{H}_{t_1}] \Big|_{x=M_{t_1}} \right].$$

Remark 4.4 If $\widehat{\mathbb{E}}[|M_t|^p] < \infty$ for all $p > 0$ and t , then $\widehat{\mathbb{E}}[(|M_t|^p - c)^+] \rightarrow 0$ as $c \rightarrow \infty$, and so (4.1) is satisfied. Further, if $\widehat{\mathbb{E}}$ is countably sub-additive, then the condition (4.1) is satisfied. The G -expectation space considered in Xu and Zhang (2009, 2010) is complete and so the sub-linear expectation is countably additive, and (4.1) is satisfied.

For proving Theorem 4.1 we need two more lemmas. The first one gives the exceptional inequality of the martingales.

Lemma 4.1 *Suppose that the operators $\widehat{\mathbb{E}}_{n,k}$ satisfy (a)-(d), $\{Z_{n,k}; k = 1, \dots, k_n\}$ is an array of random variables such that $Z_{n,k} \in \mathcal{H}_{n,k}$ and $\widehat{\mathbb{E}}[Z_{n,k}^2] < \infty$, $k = 1, \dots, k_n$. Assume that $\widehat{\mathbb{E}}[Z_{n,k}|\mathcal{H}_{n,k-1}] \leq 0$, $k = 1, \dots, k_n$. Then for all $x, y, A > 0$*

$$\begin{aligned} \mathbb{V}\left(\sum_{k=1}^{k_n} Z_{n,k} \geq x\right) &\leq \mathbb{V}\left(\max_{k \leq k_n} Z_{n,k} \geq y, \sum_{k=1}^{k_n} \widehat{\mathbb{E}}[Z_{n,k}^2|\mathcal{H}_{n,k-1}] \geq A\right) \\ &\quad + \exp\left\{-\frac{x^2}{2(xy+A)}\left(1 + \frac{2}{3}\ln\left(1 + \frac{xy}{A}\right)\right)\right\}. \end{aligned} \quad (4.2)$$

Proof. Let $X_k = Z_{n,k} \wedge y$. Then $Z_{n,k} - X_k = (Z_{n,k} - y)^+ \geq 0$. Denote $\sigma_{n,k}^2 = \widehat{\mathbb{E}}[Z_{n,k}^2|\mathcal{H}_{n,k-1}]$, $\delta_k = \sum_{i=1}^k \sigma_{n,i}^2$, $k = 1, \dots, k_n$. Let $f(x)$ be a function with bounded derivative such that $I\{x \leq A\} \leq f(x) \leq I\{x \leq A + \epsilon\}$. Let $Y_k = X_k f(\delta_k)$, $T_k = \sum_{i=1}^k Y_i$. Then $\widehat{\mathbb{E}}[Y_k|\mathcal{H}_{n,k-1}] \leq f(\delta_k)\widehat{\mathbb{E}}[Z_{n,k}|\mathcal{H}_{n,k-1}] \leq 0$, $\widehat{\mathbb{E}}[Y_k^2|\mathcal{H}_{n,k-1}] \leq f^2(\delta_k)\widehat{\mathbb{E}}[Z_{n,k}^2|\mathcal{H}_{n,k-1}] = f^2(\delta_k)\sigma_{n,k}^2$. Denote $\delta_k^* = \sum_{i=1}^k f^2(\delta_i)\sigma_{n,i}^2$. It follows that for any $x, y, A > 0$,

$$\mathbb{V}\left(\sum_{k=1}^{k_n} Z_{n,k} \geq x\right) \leq \mathbb{V}\left(\max_{k \leq k_n} Z_{n,k} \geq y, \delta_{k_n} > A\right) + \mathbb{V}(T_{k_n} \geq x).$$

For any $t > 0$, by noting

$$e^{tY_k} = 1 + tY_k + \frac{e^{tY_k} - 1 - tY_k}{Y_k^2} Y_k^2 \leq 1 + tY_k + \frac{e^{ty} - 1 - ty}{y^2} Y_k^2,$$

we have

$$\widehat{\mathbb{E}}[e^{tY_k}|\mathcal{H}_{n,k-1}] \leq 1 + \frac{e^{ty} - 1 - ty}{y^2} \widehat{\mathbb{E}}[Y_k^2|\mathcal{H}_{n,k-1}] \leq \exp\left\{\frac{e^{ty} - 1 - ty}{y^2} f^2(\delta_k) \sigma_{n,k}^2\right\}.$$

It follows that

$$\begin{aligned} \widehat{\mathbb{E}}\left[\exp\left\{-\frac{e^{ty} - 1 - ty}{y^2} \delta_k^*\right\} e^{tT_k}\right] &= \widehat{\mathbb{E}}\left[\exp\left\{-\frac{e^{ty} - 1 - ty}{y^2} \delta_k^*\right\} e^{tT_{k-1}} \widehat{\mathbb{E}}[e^{tY_k}|\mathcal{H}_{n,k-1}]\right] \\ &\leq \widehat{\mathbb{E}}\left[\exp\left\{-\frac{e^{ty} - 1 - ty}{y^2} \delta_{k-1}^*\right\} e^{tT_{k-1}}\right] \leq \dots \leq 1. \end{aligned}$$

Note $\delta_k^* \leq A + \epsilon$. We have

$$\widehat{\mathbb{E}}\left[\exp\left\{-\frac{e^{ty} - 1 - ty}{y^2} (A + \epsilon)\right\} e^{tT_k}\right] \leq 1.$$

Hence

$$\mathbb{V}(T_{k_n} \geq x) \leq e^{-tx} \widehat{\mathbb{E}}[e^{tT_{k_n}}] \leq e^{-tx} \exp\left\{\frac{e^{ty} - 1 - ty}{y^2} (A + \epsilon)\right\}.$$

Choosing $t = \frac{1}{y} \ln \left(1 + \frac{xy}{A+\epsilon}\right)$ yields

$$\mathbb{V}(T_{k_n} \geq x) \leq \exp \left\{ \frac{x}{y} - \frac{x}{y} \left(\frac{A+\epsilon}{xy} + 1 \right) \ln \left(1 + \frac{xy}{A+\epsilon} \right) \right\}.$$

Applying the elementary inequality

$$\ln(1+t) \geq \frac{t}{1+t} + \frac{t^2}{2(1+t)^2} \left(1 + \frac{2}{3} \ln(1+t)\right)$$

yields

$$\left(\frac{A+\epsilon}{xy} + 1 \right) \ln \left(1 + \frac{xy}{A+\epsilon} \right) \geq 1 + \frac{xy}{2(xy+A+\epsilon)} \left(1 + \frac{2}{3} \ln \left(1 + \frac{xy}{A+\epsilon} \right) \right).$$

(4.2) is proved by letting $\epsilon \rightarrow 0$. \square

Lemma 4.2 *Suppose that the operators $\widehat{\mathbb{E}}_t$ satisfy (i)-(iv), M_t is a martingale in $(\Omega, \mathcal{H}, \mathcal{H}_t, \widehat{\mathbb{E}})$ such that (IV') is satisfied and $\widehat{\mathbb{E}}[(M_t - M_s)^2 | \mathcal{H}_s] \leq \sigma^2(t-s)$ for all $t > s \geq 0$, where σ is a positive constant. Then*

$$\mathbb{V}(M_t - M_s \geq x) \leq \exp \left\{ -\frac{x^2}{2\sigma^2(t-s)} \right\}, \text{ for all } t > s \geq 0, x \geq 0. \quad (4.3)$$

In particular, for any $p > 0$,

$$C_{\mathbb{V}} \left([(M_t - M_s)^+]^p \right) \leq c_p \sigma^p (t-s)^{p/2}. \quad (4.4)$$

Proof. Let $s = t_0 < t_1 < \dots < t_k = t$ be a partition of $[s, t]$ with $t_i - t_{i-1} < \delta$. Note $\widehat{\mathbb{E}}[M_{t_i} - M_{t_{i-1}} | \mathcal{H}_{t_{i-1}}] = 0$ and $\widehat{\mathbb{E}}[(M_{t_i} - M_{t_{i-1}})^2 | \mathcal{H}_{t_{i-1}}] \leq \sigma^2(t_i - t_{i-1})$. So, $\sum_{i=1}^k \widehat{\mathbb{E}}[(M_{t_i} - M_{t_{i-1}})^2 | \mathcal{H}_{t_{i-1}}] \leq \sigma^2(t-s)$. By Lemma 4.1,

$$\begin{aligned} & \mathbb{V}(M_t - M_s \geq x) \\ & \leq \mathbb{V} \left(\max_i (M_{t_i} - M_{t_{i-1}}) \geq y \right) \\ & \quad + \exp \left\{ -\frac{x^2}{2(xy + \sigma^2(t-s))} \left(1 + \frac{2}{3} \ln \left(1 + \frac{xy}{\sigma^2(t-s)} \right) \right) \right\} \\ & \leq \mathbb{V}(w_t(M, \delta) \geq y) + \exp \left\{ -\frac{x^2}{2(xy + \sigma^2(t-s))} \left(1 + \frac{2}{3} \ln \left(1 + \frac{xy}{\sigma^2(t-s)} \right) \right) \right\}. \end{aligned}$$

By letting $\delta \rightarrow 0$ and then $y \rightarrow 0$, we conclude (4.3). Finally for $p > 0$,

$$\begin{aligned} C_{\mathbb{V}} \left([(M_t - M_s)^+]^p \right) & \leq \int_0^\infty \mathbb{V} \left(M_t - M_s \geq x^{1/p} \right) dx \\ & \leq \sigma^p (t-s)^{p/2} \int_0^\infty \exp \left\{ -\frac{x^{2/p}}{2} \right\} dx \leq c_p \sigma^p (t-s)^{p/2}. \quad \square \end{aligned}$$

Proof of Theorem 4.2. Let W_t be a G-Brownian motion in a sub-linear expectation $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{\mathbb{E}})$. It is sufficient to show that for any $0 < t_1 < \dots < t_d$ and $\varphi \in C_{b,Lip}(\mathbb{R}_d)$

$$\widehat{\mathbb{E}}[\varphi(M_{t_1}, \dots, M_{t_d})] = \tilde{\mathbb{E}}[\varphi(W_{t_1}, \dots, W_{t_d})]. \quad (4.5)$$

Actually, if (4.5) holds for any $\varphi \in C_{b,Lip}(\mathbb{R}_d)$, then $C_{\mathbb{V}}(|M_t|^p) < \infty$ for any t and $p > 0$ as discussed at the end of Section 1. So $\widehat{\mathbb{E}}[|M_t|^p] \leq C_{\mathbb{V}}(|M_t|^p) < \infty$ by the assumption (4.1). And then we can extend φ from $C_{b,Lip}(\mathbb{R}_d)$ to $C_{l,Lip}(\mathbb{R}_d)$ by an elementary argument. Now, without loss of generality, we assume $0 < t_1 < \dots < t_d \leq 1$. Suppose that (I)-(IV) are satisfied. Let

$$k_n = 2^n, \quad Z_{n,k} = W_{k/2^n} - W_{(k-1)/2^n}, \quad \mathcal{H}_{n,k} = \mathcal{H}_{k/2^n}, \quad k = 1, \dots, k_n.$$

Then $\widehat{\mathbb{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}] = \widehat{\mathcal{E}}[Z_{n,k} | \mathcal{H}_{n,k-1}] = 0$,

$$\widehat{\mathbb{E}}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}] = \frac{\bar{\sigma}^2}{2^n}, \quad \widehat{\mathcal{E}}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}] = \frac{\underline{\sigma}^2}{2^n}.$$

Hence it is easily seen that the sequence $\{Z_{n,k}, \mathcal{H}_{n,k}\}$ satisfy the conditions (2.3), (2.4) and (2.17) with $\rho = \bar{\sigma}^2$, $r = \underline{\sigma}^2/\bar{\sigma}^2$. Further, by the assumption (IV),

$$\sum_{k=1}^{k_n} \widehat{\mathbb{E}}[|Z_{n,k}|^{2+\delta}] = \sum_{k=1}^{2^n} o\left(\frac{1}{2^n}\right) \rightarrow 0.$$

So, the Lindeberg condition (2.1) is satisfied. Let $W_n(\cdot)$ be defined as in (2.18). By Theorem 4.1, $W_n \xrightarrow{d} W$. It follows that $(W_n(t_1), \dots, W_n(t_d)) \xrightarrow{d} (W_{t_1}, \dots, W_{t_d})$. On the other hand,

$$|W_n(t) - M_t| \leq \left| M_{([2^n t]+1)/2^n} - M_t \right| + \left| M_t - M_{[2^n t]/2^n} \right| \xrightarrow{\mathbb{V}} 0.$$

So, (4.5) holds for all $\varphi \in C_{b,Lip}(\mathbb{R}_d)$.

Finally, Suppose that (I)-(III) and (IV') are satisfied. Note that both M_t and $-M_t$ are martingales, and $\widehat{\mathbb{E}}[(M_t - M_s)^2 | \mathcal{H}_{t_{i-1}}] = \bar{\sigma}^2(t - s)$. By Lemma 4.2,

$$C_{\mathbb{V}}(|M_t - M_s|^p) \leq c_p \bar{\sigma}^p(t - s)^{p/2}.$$

By the assumption (4.1), $\widehat{\mathbb{E}}[|M_t - M_s|^3] \leq c \bar{\sigma}^3(t - s)^{3/2} = o(t - s)$ as $t - s \rightarrow 0$. So (IV) is satisfied and the proof is now completed. \square

References

- [1] Bayraktar, E. and Munk, A. (2016), An α -stable limit theorem under sublinear expectation, *Bernoulli*, **22**(4), 2548-2578
- [2] Billingsley, P. (1968), *Convergence of Probability Measures*, Wiley, New York.
- [3] Denis, L. and Martini, C. (2006), A theoretical framework for the pricing of contingent claims in the presence of model uncertainty, *Ann. Appl. Probab.*, **16**(2): 827-852.
- [4] Denis, L., Hu, M., Peng, S. (2011), Function spaces and capacity related to a sublinear expectation: application to G-Brownian Motion Pathes, *Potential Anal*, **34**:139-161. arXiv:0802.1240v1 [math.PR].
- [5] Doob, J. L. (1953), *Stochastic Processes*, John Wiley & Sons, New York.
- [6] Gilboa, I. (1987), Expected utility theory with purely subjective non-additive probabilities, *J. Math. Econom.*, **16**: 65-68.
- [7] Hall, P. and Heyde, C. C. (1980). *Martingale Limit Theory and its Applications*. Academic Press, New York.
- [8] Hu, M. S. and Li, X. J. (2014), Independence under the G -expectation framework, *J. Theor. Probab.*, **27**: 1011-1020.
- [9] Li, M. and Shi, Y.F. (2010), A general central limit theorem under sublinear expectations, *Science in China Ser. A*, **53**(8): 1989-1994.
- [10] Lin, Q. (2013), General martingale characterization of G-Brownian motion, *Stochastic Analysis and Applications*, **31**: 1024C1048.
- [11] Marinacci, M. (1999), Limit laws for non-additive probabilities and their frequentist interpretation, *J. Econom. Theory*, **84**: 145-195.
- [12] Peng, S. (1997), BSDE and related g -expectation, *Pitman Research Notes in Mathematics Series*, **364**: 141-159.
- [13] Peng, S. (1999), Monotonic limit theorem of BSDE and nonlinear decomposition theorem of Doob-Meyer type, *Probab. Theory Related Fields*, **113**: 473-499.
- [14] Peng, S. (2006), G -expectation, G -Brownian motion and related stochastic calculus of Ito type. In: (Benth F E, et al. eds.) *Stochastic Analysis and Applications, Proceedings of the Second Abel Symposium, 2005*, pp. 541-567. New York: Springer-Verlag,
- [15] Peng, S. (2007a), Law of large numbers and central limit theorem under nonlinear expectations. arXiv: 0702358v1 [math.PR].
- [16] Peng, S. (2007b), G -Brownian motion and dynamic risk measure under volatility uncertainty. arXiv: 0711.2834v1 [math.PR].
- [17] Peng, S. (2008a), Multi-dimensional G -Brownian motion and related stochastic calculus under G -expectation, *Stochastic Process. Appl.*, **118**: 2223-2253.
- [18] Peng, S. (2008b), A new central limit theorem under sublinear expectations, Preprint: arXiv:0803.2656v1 [math.PR]

- [19] Peng, S. (2009), Survey on normal distributions, central limit theorem, Brownian motion and the related stochastic calculus under sublinear expectations, *Science in China Ser. A*, **52**(7): 1391-1411.
- [20] Pollard, D. (1984), *Convergence of Stochastic Processes*, Springer-Verlag, New-York.
- [21] Xu, J. and Zhang, B. (2009), Martingale characterization of G-Brownian motion, *Stochastic Process. Appl.*, **119**: 232-248.
- [22] Xu, J. and Zhang, B. (2010), Martingale property and capacity under G-Framework, *Elect. J. Probab.*, **15**:2041C2068.
- [23] Zhang, L. X. (2015a), Rosenthal's inequalities for independent and negatively dependent random variables under sublinear expectations with applications, *Science in China Math.*, **59** (4):751-768.
- [24] Zhang, L. X. (2015b), Donsker's invariance principle under the sublinear expectation with an application to Chung's law of the iterated logarithm, *Communications in Math. Stat.*, **3**(2): 187-214.